Filters are the first order means by which signals are processed in electronic circuits. In advanced classes, filters can be designed to perform very sophisticated operations—the kinds of filters in your cell phones, for instance, can extract tiny signals from under huge noise backgrounds—but at this point our goal is to learn how to do things like remove unwanted noise or enhance transmission of desired signal ranges.

**Input and Output**

Transfer functions and filters are always considered in the context of an input and output. The signal is being processed in some way by the circuit. Often, then input will come from a source, or a Thevenin-equivalent network of some sort, but it can also be left unspecified, as just a dangling port (typically on the left hand side of the page).

Now recall that $v_{IN}$, the actual sinusoidal steady state voltage, is going to be expressed as the real part of a complex time-dependent phasor:

$$v_{IN} = V_{in} \cos(\omega t + \phi)$$
$$= \Re[\tilde{V_{in}} e^{j\omega t}]$$

$$v_{OUT} = V_{out} \cos(\omega t + \phi + \Delta \phi)$$
$$= \Re[\tilde{V_{out}} e^{j\omega t}]$$

$$\tilde{V_{out}} = H(j\omega) \tilde{V_{in}}$$

For example, suppose you are asked to find the transfer function for the filter circuit given below.
\[ v_{IN} = V_{in} \cos(\omega t) = \Re\left[V_{in}e^{j\omega t}\right] \]

\[ v_{OUT} = \Re\left[\tilde{V}_{out}e^{j\omega t}\right] = |\tilde{V}_{out}| \cos(\omega t + \angle \tilde{V}_{out}). \]

The transfer function is \( H(j \omega) \) where \( \tilde{V}_{out} = H(j \omega) V_{in} \). We’re just writing this all out with the real parts here because the concepts from sinusoidal steady state are still new. In practice, this is all implicit in the statement of the problem. \(^1\)

The concept of a transfer function is applicable in situations where the inputs are all in a sinusoidal steady state, i.e. any transients (e.g. impulse or step responses) have decayed away long since. In such a scenario, the circuit elements are all linear, which means we should proceed by modeling this circuit as a DC circuit.

\[ \Rightarrow \tilde{V}_{out} = \left(\frac{Z_C}{R + Z_C}\right) \tilde{V}_{in} \]

\[ \Rightarrow H(j \omega) = \frac{1/(j \omega C)}{R + 1/(j \omega C)} \]

\( = \frac{1}{RC} \left(\frac{1}{j \omega + \frac{1}{RC}}\right) \), and, if we define \( \omega_b \equiv 1/RC \),

\[ = \frac{\omega_b}{j \omega + \omega_b} \]

In principle, we’re done, but the problem is that the complex number itself has no meaning in the real world... it is the amplitude and phase of it that carry the meaning. The amplitude tells us about the

\(^1\) We’ve chosen \( V_{in} \) to be real because we have the liberty of selecting an arbitrary input phase... our circuit will then modify by adding to that phase (or subtracting from it). If we had a multi-stage filter, or some other way in which the problem was posed, we might not be at that liberty, in which case we would express \( V_{in} \) with a tilde above it, as \( \tilde{V}_{in} \).
scaling of the magnitude of the signal, the phase about its phase shift. So we should calculate the magnitude and phase of $H$.

$$|H(j\omega)| = \frac{|\tilde{V}_{out}|}{|\tilde{V}_{in}|} = \frac{1}{RC\sqrt{\omega^2 + \omega_b^2}},$$

which gives the amplitude scaling.

$$\angle H(j\omega) = \angle \tilde{V}_{out} - \tilde{V}_{in} \equiv \Delta \phi,$$

which gives the phase shift.

$$= \angle \left( \frac{1}{\omega_b} \frac{1}{j\omega + \omega_b} \right)$$

$$= \angle \left( \frac{1}{j\omega + \omega_b} \right)$$

$$= -\angle (j\omega + \omega_b)$$

$$= -\text{atan2}(\omega, \omega_b)$$

(4)

where we have used the notation \text{atan2} to define the arctangent function unambiguously. \text{atan2}(y, x) is defined such that the first argument represents the $y$ (or imaginary) coordinate, and the second argument represents the $x$ (or real) coordinate. This removes any possible ambiguity in which quadrant the angle resides in.

**What are the important frequency ranges?**

It is useful to look at approximations to the transfer function in the limit of large frequency, small frequency, or intermediate frequency. When performing approximations, however, one must always be careful to ask “small or large with respect to what?” In this case the denominator of the transfer function contains a clue as to how to choose the limits of the frequency. The denominator is proportional to $j\omega + \omega_b$ which has a purely imaginary component $j\omega$ and a purely real component $\omega_b$ (remember $\omega_b = 1/RC^2$). If $\omega >> \omega_b$, the denominator will be almost entirely imaginary, while if $\omega <<< \omega_b$ it will be almost entirely real. In between, when $\omega \sim \omega_b$, the complex characteristic of the denominator may be important. It may be hard to visualize what is meant by “almost real” or “almost imaginary.” Imagine what the phasor of a number like $100 + 0.1j$ looks like? It will point along the real axis... similarly $0.1 + 100j$ will point along the imaginary axis. This is what we mean by “almost real” or “almost imaginary.”

From this argument, we will look at three frequency ranges: (1) $\omega <<< \omega_b$; (2) $\omega = \omega_b$; and (3) $\omega >> \omega_b$. We can use these regimes to make a table that nicely illustrates the response of the system at low, intermediate, and high frequencies.
The traditional way to understand and visualize a filter response is with a “Bode plot.” A Bode plot compares the transfer function (expressed in decibels, i.e. $20 \log |H|$) as a function of the frequency plotted on a log axis. By inspection of the high and low frequency terms in the table above, one can see that in these limits the axes are linear, with slopes of 0 (for the low-frequency limit in this case) and 20 (in the high-frequency limit). The precise units of the slopes are decibels (the unit of the $y$ axis) per decade (because the $x$ axis multiplies frequency by 10 for each unit it progress along the logarithm). The filter slope is thus $-20$ decibels per decade at high frequencies.

![Bode plot](image)

**Figure 2**: Bode amplitude plot where we have chosen $\omega_b = 1000$ Hz (note, $\omega_b$ normally has units of rad/sec, thus we have converted to Hz here).

### 2nd-Order Bandpass Filter Example

Some circuits have really interesting and more complicated transfer functions that can be used to perform more complex operations. One particularly useful filter is a bandpass filter. It is a 2nd-order filter—called that because it requires two energy-storage elements. It is also called a resonant filter because it uses resonance (the low-loss alternation of energy storage between an inductor and capacitor) to operate.
Figure 3: Bode phase plot where we have chosen $\omega_b = 1000\text{Hz}$.

It is actually pretty possible to see just by inspection why this circuit is a "band pass" i.e. passes only a certain range of frequencies. If you imagine how it behaves at low frequencies, the capacitor has an infinite impedance ($Z_C = 1/j\omega C$), while the inductor has zero impedance, so the inductor basically shorts the output to ground and so of course $H$ will be small at low frequencies.

It is also possible to see by inspection what happens at high frequencies: if $\omega$ is large, the inductor impedance will be large ($Z_L = j\omega L$), but the capacitor impedance will be small, (because it goes like $1/\omega$), so the output is still shorted out by the capacitor and $H$ will be small at high frequencies.

Now what about at intermediate frequencies? Well, one interesting frequency in an L-C circuit (like the one we have here) is its natural response frequency $\omega_n$, $\omega_n = 1/\sqrt{LC}$. Substituting this frequency in for $\omega$, we can find that at this frequency $Z_C = -j\sqrt{L/C}$ and the inductor impedance $Z_L = j\sqrt{L/C}$. These are equal in magnitude but opposite in sign. Thus, putting them in parallel, as they appear in the circuit, the conductances are also equal and opposite and add to produce a net zero conductance! That’s right: the combination act like an open circuit in the sinusoidal steady state at the natural (or “resonant”) frequency.

The transfer function $H(j\omega) = V_{out}/V_{in}$ is also not too hard to
calculate. Redrawing the circuit, just to make the formation even easier to visualize:

![Redrawn circuit diagram](image)

The algebra might feel a bit tricky if you’re not yet quite comfortable with complex numbers. Practice will pay off here.

\[
H(j\omega) = \frac{Z_C/Z_L}{R + Z_C/Z_L}
\]

\[
= \frac{Z_CZ_L}{R + Z_CZ_L}
\]

\[
= \frac{Z_CZ_L}{RZ_C + RZ_L + Z_CZ_L}
\]

\[
= \frac{L/C}{j\omega + j\omega R + \frac{1}{C}}
\]

\[
= \frac{j\omega/RC}{1 + \frac{\omega^2}{RC} - \omega^2}
\]

\[
= \frac{2j\omega\alpha}{\omega^2 + 2j\omega\alpha - \omega^2}
\]

\[
= \frac{2j\omega\alpha}{(j\omega - \alpha)(j\omega + \alpha)}
\]

For simplicity, we will only look at the amplitude response of this filter, not its phase response. That can be found by taking the magnitude of the expression above.

\[
|H(j\omega)| = \frac{2\omega\alpha}{\sqrt{(\omega^2 - \omega^2)^2 + 4\omega^2\alpha^2}}
\]

A lot of intuition can be extracted quickly from this expression by first plotting it. Of course, adhering to the Bode plot convention, we’ll plot \(20 \log H\) rather than just \(H\).

From this plot, we see that the response is very strongly peaked near \(\omega_0\). It turns out this behavior is characteristic of such a function whenever \(\omega_0 \gg \alpha = 1/(2RC) = 1/(2\tau)\). This situation is known as an **underdamped** scenario, because it matches the underdamped time-domain scenario—namely the period of the natural oscillation frequency is much shorter than the timescale of energy decay in the system. In this limit, we can make an approximation for the transfer function around \(\omega \approx \omega_0\) that adds a lot of insight without too much algebra. For our purpose, we’ll let \(\omega = \omega_0 + \Delta\) where \(\Delta \ll \omega_0\) is the
detuning of the drive frequency relative to the resonant frequency. In this limit, the first expression in the denominator \((\omega^2 - \omega_0^2)\) can be simplified to \(\approx 2\Delta\omega_0\). In this case:

\[
|H(j\omega)| \approx \frac{2\omega\alpha}{\sqrt{4\Delta^2\omega_0^2 + 4\omega^2\alpha^2}} \\
\approx \frac{\alpha}{\sqrt{\Delta^2 + \alpha^2}} \tag{7}
\]

We’ve skipped a fair amount of grungy algebra here, including some subtle approximation steps (you have to keep throwing out higher-order terms in \(\Delta\omega/\omega_0\) and \(\alpha/\omega_0\)). You may want to spend a bit of time making sure you can do these steps, and ask for help if you get stuck.

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**Figure 4:** Bode amplitude plot where we have chosen \(\omega_0 = 1000\) Hz and \(\alpha = 10\) Hz. Resonant filters have this very characteristic “peaky” characteristic (although the peak can point down as well as up).

**Figure 5:** Linear-scale amplitude plot where we have chosen \(\omega_0 = 1000\) Hz and \(\alpha = 10\) Hz. Resonant filters have this very characteristic “peaky” characteristic (although the peak can point down as well as up). The linear scale illustrates how sharp resonances can be. The peak doesn’t quite get to 1 (see text) because sampling error in the plotting algorithm.

\(^3\) We’ll use Taylor series expansions pretty liberally in the algebra—this could be confusing, so keep it in mind as your review.
This deceptively simple expression yields tremendous insight into the nature of a resonance. Notice, for example that when $\Delta = 0$, $|H(j\omega)|$ is maximal and $\approx 1$. Also notice that when $\Delta = \pm \alpha$, $|H(j\omega)| \approx \frac{1}{\sqrt{2}}$, i.e. is reduced by a factor of $\approx 0.7$ which, when we take $20\log 0.7 \approx -3$ dB.

The consequence of this function is a peakiness, a tendency for the amplitude to be enhanced around the center. The width of the peaked region can be described as the distance between the two $-3$ dB points, i.e. $2\alpha$ relative to the center frequency $\omega_0$. The inverse of this peakiness is assigned the value $Q = \omega_0/(2\alpha)$ and is an important parameter in understanding 2nd-order filters such as bandpass and notch filters.

Some exercises for yourself to see if you know this material

(1) There are only a few possible filters out there, voltage source driving a voltage divider topologies with resister and either an inductor or a capacitor (alternating location), current source driving a current divider topology with resistor and either a capacitor or inductor... try to work out the types of filters each of these circuits are.

Some Useful Videos to Watch

If you’re looking for some weekend youtube to watch that can also educate you about this subject, check out a few videos I’ve made:

(1) https://youtu.be/yd9COpkdlNM Understanding Slopes of Bode Plots and Filter Responses

(2) https://youtu.be/QP1-9c7tDyg High-pass filter transfer function and Bode Plots