Circuits for Exercises  Use the circuits shown in figure 1 below for exercises 1 through 5.

Exercise 1
Exercise 1

\( V_{IN}(t) - R_1 i_L(t) - V_L(t) - V_c(t) = 0 \) by KVL

\( i_L(t) = \frac{\dot{V}_c(t) + V_c(t)}{R_2} = \frac{c_1 dV_c(t)}{dt} + \frac{V_c(t)}{R_2} \)

\( V_L(t) = L \frac{dI(t)}{dt} = L \frac{d}{dt} \left[ \frac{c_1 dV_c(t) + V_c(t)}{R_2} \right] \)

\( V_{OUT}(t) = V_c(t) \)

(1), (2), (3), and (4) give us

\[
\frac{V_{IN}(t)}{L} = \frac{d^2}{dt^2} V_{OUT}(t) + \frac{L + R_1 R_2}{R_2 LC} \frac{d}{dt} V_{OUT}(t) + \frac{R_1 R_2}{LC^2} V_{OUT}(t)
\]

\( i_{C_1} - i_{C_2} - i_{R_1} = 0 \) by KCL

\[
C_1 \frac{dV_{C_1}}{dt} - C_2 \frac{dV_{C_2}}{dt} - \frac{d}{dt} \left[ V_{IN} - V_{C_2} - i_{C_2} R_2 \right] = 0
\]

\[
C_1 \frac{d}{dt} \left[ V_{IN} - V_{C_2} - i_{C_2} R_2 \right] = C_2 \frac{dV_{C_2}}{dt} - \frac{V_{C_2} + V_{C_2}}{R_1} R_2 i_{C_2} = 0
\]

\[
C_1 \frac{dV_{IN}}{dt} = C_1 \frac{d}{dt} V_{C_2} - C_2 \frac{d}{dt} V_{C_2} - C_2 \frac{d}{dt} i_{C_2} R_2 - C_2 \frac{d}{dt} i_{C_2} R_2 - \frac{V_{C_2}}{R_1} R_2 i_{C_2} = 0
\]

\[
C_1 \frac{dV_{IN}}{dt} = \frac{V_{OUT}}{R_2} + \frac{C_1 i_{C_2}}{C_2} + C_2 i_{C_2} R_2 + \frac{C_2 i_{C_2}}{C_2} + \frac{1}{R_2} \left( \frac{V_{C_2} R_2}{R_1} \right) i_{C_2} = 0
\]

\[
i_{C_2} = \frac{V_{OUT}}{R_2}
\]

\[
C_1 \frac{d^2 V_{IN}}{dt^2} = \frac{C_1}{R_1 R_2} V_{OUT} + \frac{C_1}{R_2} \frac{d}{dt} V_{OUT} + \frac{C_1}{R_2} \frac{d}{dt} V_{OUT} + \frac{1}{R_2} \frac{d}{dt} V_{OUT} + \frac{1}{R_2} \frac{d}{dt} V_{OUT}
\]

\[
\frac{d^2}{dt^2} V_{IN} = \frac{d^2}{dt^2} V_{OUT} + \left( \frac{1}{R_1 R_2} + \frac{1}{R_2} + \frac{1}{C_2} \right) \frac{d}{dt} V_{OUT} + \frac{1}{R_1 R_2 R_2} V_{OUT}
\]
Exercise 2

(a) Find $H(j\omega)$ for circuit (a) in figure 1.
(b) Find $H(j\omega)$ for circuit (c) in figure 1.

(c) Find $H(j\omega)$ for circuit (e) in figure 1.

(d) Find $H(j\omega)$ for circuit (f) in figure 1.
Exercise 2
Exercise 2
Exercise 3  For each of the circuits in figure 1, assume a generic step function drive. That is, assume \( v_{IN} = u(t)V_{IN} \) and \( i_{IN} = u(t)I_{IN} \).

(a) List the initial current and voltage in the reactive circuit elements before and after the input step.

(b) Use your answers to part (a) to establish the requisite initial conditions to solve the differential equation.

(c) Solve for \( v_{OUT} \) for circuit (a) in figure 1.

(d) Solve for \( v_{OUT} \) for circuit (f) in figure 1.
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Solving for initial conditions for circuit C:

$v_{out}(t^+) = V_{SN}$. Since $v_{out}(t^+) = V_{SN} - v_C(t^+) - v_C(t^+)$

The second initial condition should be $\frac{d}{dt}v_{out}(t^+)$

Recall that $i_C(t^+) = \frac{V_{SN}}{v_C(t^+)}$ and $i_C(t^+) = \frac{V_{SN}}{R_3C_2}$

Since $i_C = \frac{d}{dt}v_C$, then $\frac{d}{dt}v_C = \frac{V_{SN}}{C_{\text{voltage}}}$

And $\frac{d}{dt}v_{out} \bigg|_{t=0^+} = \frac{V_{SN}}{C_{\text{voltage}}}$

$\frac{d}{dt} \left( v_{out} - v_C - v_C(t) \right) \bigg|_{t=0^+} = 0 - \frac{V_{SN}}{R_3C_2} - \frac{V_{SN}}{R_3C_2}$

$\frac{d}{dt}v_{out}(t) = -\frac{V_{SN}}{R_3C_2}$

Exercise 3
Exercise 3

c) Assuming a lightly damped system, then \( V_{out}(t) \) should have the following general solution:

\[
V_{out}(t) = V_p(t) + V_p(t)
= V_p(t) + A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t)
\]

where \( \omega_d \), \( \omega_d \) are taken from the differential equation from exercise 9.1 for circuit (a).

We use intuition to solve for \( V_p(t) \).
Since \( C \) is open and inductor is shorted after a long time, \( V_p(t) = V_{in} \left( \frac{R_2}{R_1 + R_2} \right) \).

We use initial conditions to solve for \( A_1, A_2 \).
From part (b) of exercise 9.1, \( V_{out}(0^+) = 0 \), \( \frac{dV_{out}(t)}{dt} = 0 \).

\[
V_{out}(t) = V_{in} \left( \frac{R_2}{R_1 + R_2} \right) - \left( \frac{V_{in} R_2}{R_1 + R_2} \right) e^{-\alpha t} \cos(\omega_d t) - \left( \frac{V_{in} R_2}{R_1 + R_2} \right) e^{-\alpha t} \sin(\omega_d t)
\]

where

\[
\omega_d = \sqrt{\frac{L}{C}}
\]

\[
\alpha = \frac{L + R_1 R_2 C}{2 L R_2 C}
\]

\[
\omega_d = \sqrt{\frac{1}{\omega_0^2} - \alpha^2}
\]
Exercise 4  Assume for each circuit in figure 1 that the drive is of the form $Acos(\omega t + \phi)$. Determine the indicated output variable.
Exercise 4

\[ H(j\omega) = \frac{\frac{1}{\sqrt{LC}}}{(j\omega)^2 + j\omega \left( \frac{L+R_1R_2C}{R_2LC} \right) + \frac{R_1+R_2}{R_2LC}} \]

\[ |H(j\omega)| = \frac{\sqrt{C}}{LC} \sqrt{\left( \frac{L+R_1R_2C}{R_2LC} \right)^2 + \left( \frac{R_1+R_2}{R_2LC} - \omega^2 \right)^2} \]

\[ + H(j\omega) = -\tan^{-1} \left( \frac{\frac{L+R_1R_2C}{R_2LC} \cdot \frac{1}{\frac{R_1+R_2}{R_2LC} - \omega^2}} \right) \]

\[ V_{out}(t) = |H(j\omega)| \cdot A \cos (\omega t + \phi + \phi H(j\omega)) \]

\[ h(s) = \frac{\tilde{V}_{out}(s)}{\tilde{V}_{in}(s)} = \frac{s^2 R_1 R_2 C_1 C_2}{s^2 R_1 C_1 R_2 C_2 + s(R_1 C_1 + R_2 C_2 + R_1 C_2) + 1} \]

\[ H(j\omega) = \frac{\omega^2 R_1 R_2 C_1 C_2}{1 - \omega^2 R_1 C_1 R_2 C_2 + j\omega(R_1 C_1 + R_2 C_2 + R_1 C_2)} \]

\[ |H(j\omega)| = \frac{\omega^2 R_1 R_2 C_1 C_2}{\sqrt{(1 - \omega^2 R_1 C_1 R_2 C_2)^2 + (\omega(R_1 C_1 + R_2 C_2 + R_1 C_2))^2}} \]

\[ H(j\omega) = -\tan^{-1} \left( \frac{\omega(R_1 C_1 + R_2 C_2 + R_1 C_2)}{1 - \omega^2 R_1 C_1 R_2 C_2} \right) \]

\[ V_{out}(t) = |H(j\omega)| \cdot A \cos (\omega t + \phi + \phi H(j\omega) + \text{due to minus sign}) \]
Exercise 4
Exercise 5  For each of the circuits in figure 1, replace the sources with impulses. That is, assume $i_{IN} = Q \delta(t)$ and $v_{IN} = \Lambda \delta(t)$. Write the expression for the indicated output variable vs time for $t > 0$.

**Solution Method I**

$$\frac{\Delta}{\Delta t} \frac{d}{dt} V_{IN} U(t) = \Delta S(t)$$

Since multiplication by $\frac{\Delta}{\Delta t}$ and taking a derivative are linear operations, then we can use our solution from exercise 9.4 part C to solve for $V_{out}(t)$ when $V_{IN} = \Lambda S(t)$.

$$V_{out}(t) = \frac{\Delta}{V_{IN}} \frac{d}{dt} \left[ V_{IN} \frac{R_2}{R_1 + R_2} \left( 1 - e^{-\frac{at}{w_d}} + \frac{\alpha t}{w_d} \right) \right]$$

$$= \Delta \frac{R_2}{R_1 + R_2} \left( \frac{\alpha t}{w_d} + \frac{\alpha t}{w_c} \right) e^{\frac{-\alpha t}{w_d}}$$

$$= \Delta \frac{R_2}{R_1 + R_2} \frac{\alpha t}{w_d} e^{\frac{-\alpha t}{w_d}}$$

$$V_{out}(t) = \frac{\Delta}{LCw_d} e^{\frac{-\alpha t}{w_d}}$$

where $\alpha t = \frac{1}{\sqrt{L_1 R_2 C}}$, $\omega = \sqrt{\frac{1}{L_1 R_2 C}}$
Exercise 9.6 (Circuit C)

Method of Solving

Find step response to \( V_{in} = V_{IN}(t) \)

Then perform \( \frac{d}{dt} \) on this response to get impulse response

\[ V_{CL} = V_1 \text{ and } V_2 \]

\[ 1 \quad - \frac{V_1}{R_1} + \frac{C_2}{R_2} \frac{d}{dt} (V_{IN} - V_1) - \frac{C_1}{R_1} \frac{d}{dt} (V_1 - V_2) = 0 \]

\[ 2 \quad \frac{C_2}{R_2} \frac{d}{dt} (V_1 - V_2) - \frac{V_2}{R_2} = 0 \]

Solve for \( V_1 \) in terms of \( V_2 \) and plug into \( 1 \)

\[ V_1(t) = V_2(t) - V_2(0) + \frac{1}{R_2} \int_0^t V_2(\tau) \, d\tau + V_1(0) \]

\[ 3 \quad V_1(t) = V_2(t) + \frac{1}{R_2} \int_0^t V_2(\tau) \, d\tau \text{ when } V_2(0) = V_2(t) = 0 \]

Differential Equation

\[ \frac{d^2}{dt^2} V_{IN} = \frac{\alpha^2}{R_1 R_2 C_2} V_{IN}(t) + \frac{1}{R_1 R_2 C_2} \int_0^t V_2(\tau) \, d\tau + \frac{1}{R_1 R_2 C_2} \int_0^t \frac{d}{dt} V_2(\tau) \, d\tau \]

\[ 2 \quad \omega = \sqrt{\frac{1}{R_1 R_2 C_2} + \frac{1}{R_2 C_2}} \]

\[ \omega_0^2 = \frac{1}{R_1 R_2 C_2} \]

Solve for initial conditions

\[ V_2(0^+) = V_{IN}^+ \text{ and } V_2(0^-) = V_{IN}(t) \left[ \frac{-i}{R_1 R_2} - \frac{1}{R_2 C_2} \right] \]

from Exercise 9.4 b

Solve for Roots of Characteristic Equation

\[ S_1, S_2 = -\omega \pm \sqrt{\omega^2 - \omega_0^2} \text{ which are Real and Negative} \]

\[ V_2(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} + V_{SP}(t) \]
Exercise 9.6 (circuit C) continued

Using initial conditions, we solve for $A_1$, $A_2$

$V_{2p}(t) = 0$ since caps are open circuits

After a long period of time

$V_2(0) = A_1 + A_2 = V_{IN} \Rightarrow A_2 = (V_{IN} - A_1)$

$$\frac{dV_2}{dt}\left|_{t=0}\right. = S_1 A_1 e^{S_1 t} + S_2 A_2 e^{S_2 t} = S_1 A_1 + S_2 (V_{IN} - A_1)$$

$$A_1 = -\frac{V_{IN}}{S_1 - S_2} \left( \frac{R_1 + R_2}{R_2 C_1} + \frac{1}{R_2 C_2} + S_2 \right)$$

Now to find the impulse response

Now $V_{out}(t) = \frac{d}{dt} \left( \frac{V_{IN} A_1 e^{S_1 t} + (V_{IN} - A_1) e^{S_2 t}}{V_{IN}} \right)$

$$= \frac{S_1}{V_{IN}} (S_1 A_1 e^{S_1 t} - S_2 A_1 e^{S_2 t})$$

$$V_{out}(t) = \Delta\left[ \frac{S_1}{S_1 - S_2} \left( \frac{R_1 + R_2}{R_2 C_1} + \frac{1}{R_2 C_2} + S_2 \right) e^{S_1 t} + (S_2 + S_1) \frac{R_1 + R_2}{S_1 - S_2} e^{S_2 t} \right]$$

Where $S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$

$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$
\[
\frac{d^2}{dt^2}i_{\text{out}}(t) + \left[\frac{R_1 R_2}{L_1 L_2} + \frac{R_2}{L_1} + \frac{R_1}{L_2}\right] \frac{di_{\text{out}}(t)}{dt} + \frac{R_2}{L_1} i_{\text{out}}(t) = \ldots
\]

\[
\ldots = \frac{d}{dt} i_{\text{IN}}(t) \frac{R_2}{L_1} + \frac{R_1 R_2}{L_2} i_{\text{IN}}(t)
\]

- First, we will solve for the step response to input \(i_{\text{IN}}(t)\).
- Second, we take \(\frac{d}{dt} i_{\text{OUT}}(t)\) and \(i_{\text{OUT}}(t)\) to get the impulse response.

Since this is a second order system, the general step response is:

\[
i_{\text{OUT}}(t) = i_{\text{OUT}}(t) + A_1 e^{s_1 t} + A_2 e^{s_2 t}
\]

where \(s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}\)

For this circuit, however, \(s_1, s_2\) are both real and negative.

- From Exercise 9.4 by \(i_{\text{OUT}}(t) = 0\) and \(\frac{di_{\text{OUT}}(t)}{dt} = \frac{i_{\text{IN}} R_2}{L_2}\)

- \(i_{\text{OUT}}(t) = i_{\text{IN}}\) since \(L_2\) is a short after long periods.

\[
i_{\text{OUT}}(t) = i_{\text{IN}} + A_1 e^{s_1 t} + A_2 e^{s_2 t}
\]

Where \(A_1 = \frac{i_{\text{IN}} (s_2 + s_2)}{s_1 - s_2}\) \(A_2 = -i_{\text{IN}} \left(\frac{e^{s_1 t} + s_2}{s_1 - s_2}\right) - i_{\text{IN}}\)
\[
\frac{Q}{J_{1N}} \frac{d}{dt} i_{out}(t) = \frac{Q}{J_{1N}} s_1 R_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}
\]

\[
= Q \left( \frac{R_2 + S_2}{L_2} \right) s_1 e^{s_1 t} - Q \left[ \frac{R_2 + S_2}{L_2} \frac{1}{s_1 - s_2} + 1 \right] s_2 e^{s_2 t}
\]

Where
\[
s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}
\]
\[
s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}
\]
\[
\alpha = \frac{1}{2} \left[ \frac{R_1 L_2 + L_1 R_2}{L_1 L_2} \right]
\]
\[
\omega_0 = \frac{R_1 R_2}{L_1 L_2}
\]
\[ \Delta S(t) = \frac{\Delta}{\Delta S(t)} \]  

Since multiplying by \( \frac{\Delta}{\Delta S(t)} \) is a linear operation and circuit (3) is a linear circuit, then we can use our solution from Exercise 9.4 Part (a) to solve for \( V_{out}(t) \) when \( V_{SN} = \Delta S(t) \):

\[ V_{out}(t) = \frac{\Delta}{\Delta S(t)} \left[ V_{SN} - V_{SN} e^{-\omega t} \right] - \frac{\Delta}{\Delta S(t)} \frac{\omega}{\omega_d} V_{IN} e^{-\omega t} \sin(\omega_d t) \]

\[ V_{out}(t) = \Delta \left[ e^{-\omega t} \cos(\omega_d t) + \omega_d e^{-\omega t} \sin(\omega_d t) - e^{-\omega t} \sin(\omega_d t) \right] \]

Where:

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ \omega = \frac{R_1 R_2}{(R_1 + R_2) L^2} \]

\[ W_d = \frac{-\omega^2 - \omega_0^2}{\omega_0} \]

\[ V_{out}(t) = \Delta \frac{W_d - \omega_0^2}{\omega_d} e^{-\omega t} \sin(\omega_d t) \]

\[ = \Delta \frac{\omega_0^2}{\omega_d} e^{-\omega t} \sin(\omega_d t) \]

\[ V_{out}(t) = \frac{\Delta}{LCW_d} e^{-\omega t} \sin(\omega_d t) u(t) \]
Problem 1  In one of the in-class demos, we showed how a long cable could contribute enough capacitance to distort and filter a waveform. Scope probes are designed to address this problem by using a capacitive voltage divider composed of the variable capacitor $C$ and the 8pF capacitor shown below in figure 2. $C_{cable}$ models the added capacitance contributed by the cable in the demo.

![Figure 2: Problem 9.1](image)

The experiment uses a 3ft long 50Ω coax cable (Here, the 50Ω refers to the transmission line impedance. In case you are interested, that is the ratio of the amplitude of $v_{IN}$ to $i_{IN}$ when the cable is driven by high frequency sinusoidal input voltage $v_{IN}$.) However, the 50Ω impedance is not relevant for this problem. The only point of relevance is the cable parasitic capacitance, which is $30\text{pF}/\text{ft}$. Thus the 3ft long coax cable adds 90pF of capacitance in parallel with the 8pF scope capacitance.

(a) What should $C$ be such that $H(j\omega) = \frac{v_{SCOPE}(j\omega)}{v_{IN}(j\omega)}$ has no frequency dependence? What is the resulting $H(j\omega)$ with the chosen $C$?

(b) Prove that this circuit will faithfully reproduce any input waveform, $v_{IN}$ at the scope, i.e. that $v_{SCOPE}(t) = A v_{IN}(t)$ for any $v_{IN}$ and a constant $A$.

(c) You have a new kind of scope that plots current, $i_{SCOPE}$ rather than voltage, $v_{SCOPE}$, at its output port. Design a probe that will faithfully reproduce the shape of $i_{IN}(t)$ on the scope, removing the filtering influence of $C_{scope}$.

![Figure 3: Problem 1](image)
Problem 1

(a)

\[ \frac{\tilde{U}_{\text{scope}}(s)}{\tilde{U}_{\text{IN}}(s)} = \frac{R_2}{S R_2 (C_1 + C_2) + 1} \]

\[ = \frac{R_2}{S R_2 (C_1 + C_2) + 1} + \frac{R_1}{S R_1 C + 1} \]

\[ = \frac{(S R_1 C + 1) R_2}{(S R_1 C + 1) R_2 + (S R_2 (C_1 + C_2) + 1) R_1} \]

Want: \[ C = \frac{1}{9}(C_1 + C_2) = 1/2 \text{pF} \]

\[ H(j\omega) \bigg|_{C=\frac{1}{9}(C_1 + C_2)} = \frac{1}{10} \]

(b)

\[ H(s) = \frac{R_2}{S R_2 (C_1 + C_2) + 1} \]

\[ = \frac{R_2}{S R_2 (C_1 + C_2) + 1} + \frac{R_1}{S R_1 C + 1} \]

\[ = \frac{R_2}{S R_2 (C_1 + C_2) + 1} + \frac{9 R_2}{S 9 R_2 C + 1} \]

\[ R_1 = 9 R_2 \]

But when \( C = \frac{1}{9}(C_1 + C_2) \), then \( 9 R_2 C + 1 = S R_2 (C_1 + C_2) + 1 \). Thus \( H(s) = \frac{R_2}{10 R_2} \frac{1}{10} \)
Problem 1

Solution #1

\[ i_{\text{scope}}(s) = \frac{1}{sC} \cdot i_{\text{IN}}(s) = \frac{C_{\text{scope}} \cdot i_{\text{IN}}(s)}{C + C_{\text{scope}}} \]

\[ i_{\text{scope}}(t) = i_{\text{IN}}(t) \cdot \frac{C_{\text{scope}}}{C + C_{\text{scope}}} \]

Since it is said in the Problem description that the \textit{DUT} presents a current waveform, then this implies that \( i_{\text{IN}} \) is independent of the Voltage across the DUT. Thus another Acceptable Solution is just a wire.

Solution #2
Problem 2  Consider an unknown circuit shown in figure 4, exhibiting the frequency response shown in figure 5.

![Figure 4: Problem 2](image1)

![Figure 5: Problem 2](image2)

(a) What is the radial resonance frequency $\omega_o$?

(b) What is the quality factor $Q$ for this filter?

(c) Assuming $v_{IN} = 3V \ast \cos(2\pi \ast 9kHz + 0.3)$, determine $v_{OUT}$.

(d) Assuming $v_{IN} = 3V \ast (1 - u(t))$, determine the damping factor $\alpha$ and the period $T$ between zero-crossings of $v_{OUT}$ for $t > 0$. $T = \frac{2\pi}{\omega_d}$ where $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$.

(e) Suggest an LRC circuit for the black box, choosing L,R,and C values appropriately.

(f) For the circuit from part(e), find $v_{OUT}(0 + \varepsilon)$ and $\frac{d}{dt}v_{OUT}(0 + \varepsilon)$, the initial conditions.

(g) Using the standard form of a step response to an LRC circuit, write an analytic expression for $v_{OUT}(t)$ for $t > 0$. 
a) \[ W_0 = 2\pi \times 10^{3} \text{ kHz} = \frac{62.8}{\text{sec}} \text{ rad} \]

b) \[ Q = \frac{f_r}{\Delta f} = \frac{10 \text{ kHz}}{1 \text{ kHz}} = 10 \]

c) \[ V_{\text{out}}(t) \approx 0.4*3V \cos(2\pi \cdot 9 \text{ kHz} \cdot t + 0.3 + 1.16 \text{ rad}) \]

used graphical methods

- to estimate \( |H(j2\pi \cdot 9 \text{ kHz})| \approx 0.4 \)
- and to estimate \( \angle H(j2\pi \cdot 9 \text{ kHz}) \approx 67^\circ = 1.16 \text{ rad} \)

d) \[ \omega = \frac{W_0}{2\pi} \quad \alpha = \frac{W_0}{2Q} = \frac{62.8}{2 \cdot 10} \text{ rad/sec} \]

\[ \alpha \approx 3.14 \times 10^2 \]

\[ W_d = \sqrt{W_0^2 - \alpha^2} \approx 62.72 \times 10^3 \text{ rad/sec} \]

\[ \tau' = \frac{2\pi}{W_d} = \frac{2\pi}{62.72} \times 10^3 \text{ rad/sec} \]

\[ \tau'' = 100.125 \times 10^{-6} \]
e) A Series LRC circuit has a similar frequency response

\[ \frac{V_{out}(s)}{V_{in}} = \frac{\frac{sRLC}{s^2 + \frac{Q^2}{L} \cdot \frac{1}{s} + \frac{1}{LC}}} \]

Where, \(2\omega = \frac{R}{L} \), \(\text{Q} = \frac{\text{V}_{max}}{\frac{R}{2L}} = \sqrt{\frac{\text{E}}{R}} \)

From frequency response, \(\omega^2 = (2\pi \times 10^6 \text{rad/s})^2 = 3.95 \times 10^{12} \text{(rad/s)^2} \)

From part (d), \(2\omega = 2\pi (31.4 \times 10^2) = 62.8 \times 10^2 \)

Solve following system for L, R, C

\[
\begin{cases}
6.28 \times 10^2 \frac{R}{L} = 2\omega \\
62.8 \times 10^2 = \frac{1}{\text{Q}} \omega_0 \\
\text{Q} = 10 = \sqrt{\frac{\text{E}}{R}} \frac{1}{L}
\end{cases}
\]

R = 1.59 m\Omega
C = 1.5 nF
L = 0.25 mH

5) \[ V_{out}(t = 0^+) = 0 \]
\[ \frac{dV}{dt} \text{ at } (t = 0^+) = -\frac{E}{R} \times 3V = -18.18 \text{ V} \]

When \(V_{in}\) goes from 3V to 0V the capacitor will want to discharge, which requires current to flow in the direction of \(i_L\) shown below. Since inductor doesn’t want current to flow, it generates

\[ i_L = 3V \times \frac{L}{R} \text{ V-sec} \]

\[ \frac{dV(t)}{dt} = -R \frac{dI}{dt} - R \frac{dV(t)}{dt} \]

3V as shown. This 3V drop in \(V_L\) is: \[ \frac{d}{dt} \text{ means } \frac{dV(t)}{dt} = -R \frac{dI}{dt} - R \frac{dV(t)}{dt} \]
9) From the frequency response of the circuit, \( Q = 10 \) implies a lightly damped system. Thus, the unit step response has the following form...

\[
V_{out}(t) = V_p(t) + A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t)
\]

\[
V_{out}(t \to \infty) = 0 \Rightarrow V_p(t) = 0
\]

\[
V_{out}(t) = A_1 e^{-\alpha t} \cos(\omega_d t) + A_2 e^{-\alpha t} \sin(\omega_d t)
\]

\[
V_{out}(t = 0^+) = 0 \Rightarrow A_1 \Rightarrow A_1 = 0
\]

\[
\frac{dV_{out}(t)}{dt} \bigg|_{t = 0^+} = -18.868 = -A_2 \alpha e^{-\alpha t} \sin(\omega_d t) + \omega_d A_2 e^{-\alpha t} \cos(\omega_d t)
\]

\[
\Rightarrow A_2 \omega_d
\]

\[
A_2 = -\frac{18.868}{\frac{62.72 \times 10^3 \text{ rad/sec}}{62.72 \times 10^3 \text{ rad/sec}}} \approx -0.3 V
\]

\[
V_{out}(t) = -0.3V e^{-\alpha t} \sin(\omega_d t)
\]

Where \( \alpha = 3.14 \times 10^{-2}, \omega_d = 62.72 \times 10^3 \text{ rad/sec} \)

from part d
Problem 3  The network shown below models an oscilloscope probe that provides a 10:1 voltage attenuation. Resistor $R_1$ is a fixed resistor in the probe, resistor $R_2$ models the input resistance of the oscilloscope, capacitor $C_1$ is a variable capacitor in the probe, and capacitor $C_2$ models the combined input capacitance of the oscilloscope and the cable between the probe and the oscilloscope. What relations are required between $R_1$, $R_2$, $C_1$ and $C_2$ so that $v_{OUT}(t) = 0.1v_{IN}(t)$ for all $\omega$? That is, what relations are required so that $V_{out} = 0.1V_{in}$ and $\phi = 0$ for all $\omega$? (Note that the value of $C_2$ is difficult to guarantee in practice due to variations in cable length and oscilloscope input capacitance, so $C_1$ is made manually adjustable in the probe.)
Exercise 10.2

\[ V_{\text{out}}(t) = \frac{1}{2} V_{\text{in}}(t) + \frac{1}{2} V_{\text{in}}(t) \cos(\omega t + \phi) \]

\[ V_{\text{in}}(t) \]

* We desire \( V_{\text{out}} = 0.1 V_{\text{in}} \) and \( \phi = 0 \) for all \( \omega \)

A * First Thoughts

1. \( \phi = 0 \) for all \( \omega \) \( \Rightarrow \) \( V_{\text{out}}(t) \) is purely real
2. \( V_{\text{out}}(t) = 0.1 V_{\text{in}}(t) \) \( \Rightarrow \) I just want a voltage divider circuit

B * Background

Voltage Dividers...

1. \( \ldots \) W/Resistors

Assuming we don't have capacitors in the circuit above, \( V_{\text{out}}(t) = \frac{R_2}{R_1 + R_2} V_{\text{in}}(t) \)

2. \( \ldots \) W/Capacitors

Assuming we don't have resistors in the circuit above, that is, we get the following circuit...

\[ V_{\text{out}}(t) = \frac{1}{c_1 + c_2} V_{\text{in}}(t) \]

\[ \frac{V_{\text{out}}(t)}{V_{\text{in}}(t)} = \frac{c_1}{c_1 + c_2} = 0.1 \text{ for all } \omega \]

Exercise 10.2 continued

* Since we want \( V_{\text{out}}(t) = 0.1 V_{\text{in}}(t) \)

then we want,

\[ \frac{c_1}{c_1 + c_2} = \frac{R_2}{R_1 + R_2} = 0.1 \]

The relation above implies we want

| i) \( R_2 = \frac{1}{9} R_1 \) |
| ii) \( c_1 = \frac{1}{9} c_2 \) |
| iii) \( c_1 R_1 = c_2 R_2 \) |

See Alternate Way of Arriving to Solution at the end of PSET Solution
Problem 4 This problem focuses on the (greatly simplified) design of crossover networks for audio speaker systems driven by a single amplifier. The purpose of these networks is to direct low-frequency signals to a low frequency (LF) speaker, mid-range signals to a mid-range (MR) speaker, and high-frequency signals to a high-frequency (HF) speaker. The interconnection of the amplifier, modeled here as a cosinusoidal voltage source, the three speakers, each modeled here as a resistor, and the three cross-over networks is shown below. *Note that this was once a final exam problem.*

The three cross-over networks $N_{LF}$, $N_{MR}$, and $N_{HF}$ may each be a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor. The topology of each network, and the values of the components in each network, are to be designed to provide the three voltage responses shown below as functions of frequency. Note that $\omega_L$ and $\omega_H$ are the frequencies at which $|v_{LF}|$, $|v_{MR}|$ and $|v_{HF}|$ fall to the value of $V_a/\sqrt{2}$. They are related by $\omega_L \ll \omega_H$ in this problem.

(a) Consider the network $N_{LF}$ for the low-frequency speaker. What type of network should be used for $N_{LF}$: a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

(b) Determine the values of the corresponding capacitance $C_{LF}$, if used, and inductance $L_{LF}$, if used, in terms of $R_{LF}$, $R_{MR}$, $R_{HF}$, $\omega_L$ and $\omega_H$.
(c) Consider the network $N_{MR}$ for the mid-range speaker. What type of network should be used for $N_{MR}$, a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

(d) Determine the values of the corresponding capacitance $C_{MR}$, if used, and inductance $L_{MR}$, if used, in terms of $R_{LF}$, $R_{MR}$, $R_{HF}$, $\omega_L$ and $\omega_H$.

(e) Consider the network $N_{HF}$ for the high-frequency speaker. What type of network should be used for $N_{HF}$, a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

(f) Determine the values of the corresponding capacitance $C_{HF}$, if used, and inductance $L_{HF}$, if used, in terms of $R_{LF}$, $R_{MR}$, $R_{HF}$, $\omega_L$ and $\omega_H$. 
Problem 10.1 Solution Summary

(A) Consider the network $N_{LF}$ for the low-frequency speaker. What type of network should be used for $N_{LF}$: a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

Answer: Single inductor network.

(B) Determine the values of the corresponding capacitance $C_{LF}$, if used, and inductance $L_{LF}$, if used, in terms of $R_{LF}, R_{MR}, R_{HF}, \omega_L$, and $\omega_H$.

Answer: $C_{LF}$ not used and $L_{LF} = \frac{R_{HF}}{\omega_L}$

(C) Consider the network $N_{MR}$ for the mid-range speaker. What type of network should be used for $N_{MR}$: a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

Answer: Series capacitor and inductor pair.

(D) Determine the values of the corresponding capacitance $C_{MR}$, if used, and inductance $L_{MR}$, if used, in terms of $R_{LF}, R_{MR}, R_{HF}, \omega_L$, and $\omega_H$.

Answer: $C_{MR} = \frac{1}{\omega_L R_{MR}}$ and $L_{MR} = \frac{R_{MR}}{\omega_H}$

(E) Consider the network $N_{HF}$ for the high-frequency speaker. What type of network should be used for $N_{HF}$: a single capacitor, a single inductor, a series capacitor and inductor, or a parallel capacitor and inductor?

Answer: Single capacitor network.

(F) Determine the values of the corresponding capacitance $C_{HF}$, if used, and inductance $L_{HF}$, if used, in terms of $R_{LF}, R_{MR}, R_{HF}, \omega_L$, and $\omega_H$.

Answer: $C_{HF} = \frac{1}{\omega_H R_{HF}}$ and $L_{HF}$ not used.
Problem 5  In this problem, a low-voltage sinusoidal source is coupled to a resistive load through an inductor-capacitor network as shown below. The role of the network is to boost the voltage at the load.

\[ \begin{align*}
&v_{\text{IN}} \\
&L \\
&C \\
&R \\
&+ v_{\text{OUT}} \\
&- \\
\end{align*} \]

(a) Derive a second-order differential equation that describes the evolution of \( v_{\text{OUT}} \) as driven by \( v_{\text{IN}} \). You need not solve the differential equation.

(b) Assume that the circuit operates in the sinusoidal steady state with \( v_{\text{IN}}(t) = V_{\text{in}}\cos(\omega t) \). Correspondingly, let \( v_{\text{OUT}}(t) \) take the form \( v_{\text{OUT}} = V_{\text{out}}\cos(\omega t + \phi) \). Determine \( \phi \), and the voltage gain \( G \) defined by \( G \equiv V_{\text{out}}/V_{\text{in}} \).

(c) For a given \( R, L \) and \( \omega \), determine the value of \( C \) that maximizes \( G \), and for this value of \( C \), determine \( G \).

(d) Suppose that \( v_{\text{IN}}(t) \) is abruptly set to zero in an attempt to remove the voltage at the load. In this case, the amplitude of the load voltage will decay in proportion to \( e^{t/\tau} \). Assuming that \( C \) is chosen to maximize \( G \) following the result from Part (c), determine \( \tau \) in terms of \( G \) and \( \omega \). Hint: can you get the time constant (as a function of \( C, L \) and \( R \)) from the differential equation found in Part (a)?

(e) In view of the results of Parts (c) and (d), what is the disadvantage of using an inductor-capacitor network to boost the voltage which excites the load?
A.) The differential Equation can be derived from the denominator of the transfer function \( \frac{\hat{V}_{\text{out}}(s)}{\hat{V}_{\text{in}}(s)} \).

\[
\frac{\hat{V}_{\text{out}}(s)}{\hat{V}_{\text{in}}(s)} = \frac{\frac{R}{sC}}{\frac{1}{sC} + \frac{R}{sC}} = \frac{\frac{R}{sC}}{\frac{R}{sC} + \frac{sL}{sC} + sRL} = R + sL + s^2 RL
\]

\[
= \frac{\frac{1}{sLC}}{s^2 + \frac{s}{RC} + \frac{1}{sLC}}
\]

\[
\frac{d^2}{dt^2} \hat{V}_{\text{out}} + \frac{d}{dt} \hat{V}_{\text{out}} \left( \frac{1}{sRC} \right) + \frac{1}{sLC} \hat{V}_{\text{out}} = \hat{V}_{\text{in}} \left( \frac{1}{sLC} \right)
\]

B.) \( \hat{V}_{\text{out}}(jw) \) = \( \frac{\frac{1}{sLC}}{\frac{1}{sLC} - w^2 + jw \frac{1}{sRC}} \) = \( H(jw) \)

\[
|H(jw)| = \frac{\frac{1}{sLC}}{\sqrt{\left(\frac{1}{sLC} - w^2\right)^2 + \left(jw \frac{1}{sRC}\right)^2}} = G
\]

\[
\phi = \circ - \tan^{-1} \left( \frac{w \frac{1}{sRC}}{\frac{1}{sLC} - w^2} \right)
\]
(c) \[ G = \frac{\frac{1}{2}c}{\sqrt{(\frac{1}{2c} - w^2)^2 + (\frac{w}{Rc})^2}} = \frac{1}{\sqrt{(1-w^2c)^2 + (\frac{w}{Re})^2}} \]

\[ G_{\text{max}} \text{ when } (1-w^2c) = 0 \Rightarrow c = \frac{1}{w^2L} \]

\[ G_{\text{max}} = \frac{1}{\frac{WL}{R}} = \frac{R}{WL} \]
D) When $V_{in}=0$, we get the following circuit:

From Part A, we know the differential equation for this undriven circuit is:

$$\frac{d^2}{dt^2} V_{out} + \frac{1}{RC} \frac{d}{dt} V_{out} + \frac{1}{L} V_{out} = 0$$

We know the solution to this circuit has a decaying factor $e^{-\alpha t}$ where $\alpha = \frac{1}{2RC}$.

$$e^{-\frac{1}{2RC}t} = e^{-\frac{1}{\tau}} \Rightarrow \tau = 2RC$$

E) Disadvantages of using an LC network to boost the voltage that excites the load:

From Part D: there is a delay between the output and input command.

From Part C: the voltage boost only happens at one frequency.
Problem 2: Op-Amps – 20%

Assume that the op-amps in this problem are ideal. For Parts (2A) and (2B), consider the circuit shown below.

(A) Assume that the inductor and capacitor are initially at rest. At $t = 0$ the voltage source steps to $V$ such that $v_{IN}(t) = Vu(t)$ where $u(t)$ is the unit step function. Determine $v_{OUT}(t)$ for $t \geq 0$.

$$v_{OUT}(t) = -\frac{V}{2LC} t^2 \quad \text{for } t \geq 0$$

a) $v_{IN} = Vu(t)$

$$i(t) = \frac{1}{L} \int_0^t v_{IN} \, dt = \frac{V}{L} t \quad t > 0$$

$$v_{OUT}(t) = -\frac{1}{C} \int_0^t i(t) \, dt = -\frac{V}{2LC} t^2 \quad t > 0$$
(B) Let \( v_{in}(t) = V \cos(\omega t) \). Further, assume that \( v_{out} \) has a zero-valued time average. Determine \( v_{out} \) in the sinusoidal steady state.

\[
v_{out}(t) = \frac{V}{\omega^2 LC} \cos(\omega t)
\]

\[ b) \ v_{in} = V \cos \omega t \]

\[
I = \frac{V}{j\omega L}
\]

\[
V_o = -\frac{V}{j\omega L} \cdot \frac{1}{j\omega C} = \frac{V}{\omega^2 LC}
\]

\[
v_{out}(t) = \text{Re} \ V_o e^{j\omega t} = \frac{V}{\omega^2 LC} \cos \omega t
\]
For Part (2C), consider the circuit shown below.

\[ v_{\text{OUT}} = 2v_{\text{IN}} \left( \frac{R_1 + R_2}{R_2} \right) \]

(C) Determine the output voltage \( v_{\text{OUT}} \).

\[ i = \frac{v^+}{R_3} = \frac{v_{\text{IN}}}{R_3} \left( \frac{R_1 + R_2}{R_2} \right) \]

\[ v_{\text{OUT}} = 2v_{\text{IN}} \left( \frac{R_1 + R_2}{R_2} \right) \]
Problem 3: Second-Order Network Transients – 20%

This problem concerns the second-order network shown below. To begin, the current source delivers a current impulse with area $Q$, the total charge delivered, such that $I(t) = Q\delta(t)$. A graph of the subsequent capacitor voltage $v$ in volts and inductor current $i$ in milliamps is also shown below as a function of time in microseconds. Use the graphical waveforms to answer the following questions.
(A) What is the approximate value of the inductance \( L \)? A numerical answer with proper units is expected.

\[
L = 1.06 \text{ mH}
\]

(B) What is the approximate value of the capacitance \( C \)? A numerical answer with proper units is expected.

\[
C = 3.73 \text{ nF}
\]

\[
\omega_0 = \sqrt{\omega_d^2 - \alpha^2}, \quad Q = \frac{\omega_0}{2\alpha} \Rightarrow \omega_d = \omega_0 \sqrt{1 + \frac{1}{4Q^2}}
\]

Because \( Q > 5 \), \( \omega_d \approx \omega_0 \)

\[
\Rightarrow Z_0 = \frac{L}{C} \approx \frac{V_{\text{peak}}}{I_{\text{peak}}} = \frac{2V}{3.75 \text{ mA}} = 533.3 \Omega
\]

\[
\frac{1}{\sqrt{LC}} = \omega_0 \approx \omega_d = \frac{2\pi}{12.5 \times 10^{-6} \text{ s}} = 5.027 \times 10^5 \text{ rad/s}
\]

\[
L = \frac{Z_0}{\omega_0} = \frac{533.3}{5.027 \times 10^5} = 1.061 \times 10^{-2} \text{ H}
\]

\[
C = \frac{1}{\omega_0 Z_0} = \frac{1}{(5.027 \times 10^5) 533.3} = 3.730 \times 10^{-9} \text{ F}
\]
(C) What is the approximate value of the resistance \( R \)? Hint: \( e^{-0.5} \approx 0.6 \). A numerical answer with proper units is expected.

\[
R = 6.7 \text{ k} \Omega
\]

At \( t = 25 \mu s \), the envelope of capacitor voltage decays approximately to the 60% of the initial voltage.

\[
\Rightarrow 0.6 = e^{-\alpha (25 \mu s)}
\]

\[
\alpha = \frac{0.5}{25 \times 10^{-6} \text{ s}} = 2 \times 10^4 \text{ s}^{-1}
\]

For parallel RLC circuits, \( \alpha = \frac{1}{2RC} \)

\[
\Rightarrow R = \frac{1}{2\alpha C} = \frac{1}{2(2 \times 10^4)(3.73 \times 10^{-9})} = 6.702 \times 10^3 \text{ } \Omega
\]

(D) What is the approximate value of the impulse charge \( Q \)? A numerical answer with proper units is expected.

Charge \( Q = 7.46 \text{ nC} \)

\[
Q S(t) = \frac{V(t)}{R} + C \frac{dV(t)}{dt} + \frac{1}{L} \int_{-\infty}^{t} V(t') \, dt'
\]

By integrating both sides from \( t = 0^- \) to \( t = 0^+ \), we obtain

\[
Q = C(V(0^+) - V(0^-)) = (3.73 \times 10^{-9}) \cdot 2 = 7.46 \times 10^{-9} \text{ C}
\]
(E) What is the approximate value of the quality factor of the network? A numerical answer with proper units is expected.

Quality Factor = 12.57

For parallel RLC circuits

\[ Q = \frac{\omega_0}{2\alpha} = \frac{1}{\sqrt{LC}} \cdot RC = R\sqrt{\frac{C}{L}} \]

\[ \Rightarrow Q = 6.7 \times 10^3 \sqrt{\frac{3.73 \times 10^{-4}}{1.06 \times 10^{-2}}} = 12.57 \]
(F) Complete each sentence in the chart below by circling the phrase that correctly replaces the question marks. In completing the last two sentences, assume that the impulse charge $Q$ remains fixed.

<table>
<thead>
<tr>
<th></th>
<th>increases</th>
<th>decreases</th>
<th>stays fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>The oscillation frequency ??? with increasing $L$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The oscillation frequency ??? with increasing $C$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The decay rate ??? with increasing $L$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The decay rate ??? with increasing $C$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The decay rate ??? with increasing $R$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The current amplitude ??? with increasing $L$.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The voltage amplitude ??? with increasing $L$.</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For parallel RLC circuits,

- oscillation frequency $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}$
- decay rate $\alpha = \frac{1}{2RC}$
- voltage amplitude is proportional to $\frac{Q}{C}$
- current amplitude is proportional to $\left(\frac{Q}{C}\right)/Z_0 = \frac{Q}{C} \sqrt{\frac{C}{L}}$
Finally, suppose that the current source is a sinusoid taking the form \( I(t) = I_0 \cos(\omega t) \), and that values of \( R, C, \) and \( L \) are unchanged. In this case, answer the following questions assuming that the network operates in the sinusoidal steady state.

(G) For a given value of \( I_0 \), approximately what value for the frequency \( \omega \) will yield the largest amplitude for the voltage \( v \)? A numerical answer with proper units is expected.

\[
\omega = 502.9 \text{ krad/s}
\]

The largest amplitude is achieved at \( \omega_0 \).

\[
\Rightarrow \omega = \frac{1}{\sqrt{LC}} = \sqrt{\frac{1}{(1.06 \times 10^{-3})(3.73 \times 10^{-9})}} = 5.029 \times 10^5 \text{ rad/s}
\]
(H) Assume that \( \omega \) is set to be that frequency found in Part (3G) and that \( I_0 = 1 \text{ mA} \). What will be the \textit{approximate} maximized amplitude of \( v \)? A numerical answer with proper units is expected.

\[ |v| = 6.7 \text{ V} \]

In parallel RLC circuits, at the resonant frequency \( \omega_0 \), both \( L \) and \( C \) cancel out each other.

\[ \Rightarrow \quad \boxed{\text{I}(t) = I_0 \cos(\omega t) \quad \uparrow \quad \text{V} \quad \downarrow \quad R} \]

\[ |v|_{\text{max}} = I_0 R = 1 \text{ mA} \cdot 6.7 \text{ k}\Omega = 6.7 \text{ V} \]
Problem 4: Sinusoidal Steady State – 20%

The circuits shown below are driven by sinusoidal voltage sources, and operate in the sinusoidal steady state with transfer function \( H(\omega) = \frac{V_{out}}{V_{in}} \) where \( V_{in} \) and \( V_{out} \) are complex voltage magnitudes. For each circuit below, circle the letter that corresponds to the plot of its transfer function magnitude (A-F), and to the plot of its transfer function phase (G-L); the plots are shown on the following page. The magnitude plots have log-log scales, and the phase plots have log-linear scales. These plots are sketches, and so you should choose one with a correct approximate shape. Further, the horizontal axes of the log-magnitude plots do not necessarily intersect the vertical axes at log |\( H | = 1. The op-amp in the last circuit below is ideal.

1. [Diagram of circuit 1]
   - Magnitude: A B C D \( \textcircled{E} \) F
   - Phase: G \( \textcircled{H} \) I J K L

2. [Diagram of circuit 2]
   - Magnitude: A B C D E F
   - Phase: G H I J \( \textcircled{K} \) L

3. [Diagram of circuit 3]
   - Magnitude: A B C D E F
   - Phase: \( \textcircled{G} \) H I J K L

4. [Diagram of circuit 4]
   - Magnitude: A B C D E F
   - Phase: G H I J K L
Problem 2 – 35%

The circuit shown below models the electrical interaction between a digital processor, its power supply, and the connection between the two. Specifically, the current source ($i_\text{p}$) models the processor, the voltage source ($v_\text{s}$) models the supply, the resistor ($R$) and inductor ($L$) model the parasitics introduced by the interconnect wiring, and the capacitor ($C$) helps filter the processor voltage ($v_\text{p}$).

Both $i_\text{p}$ and $v_\text{s}$ can be functions of time. For example, the current drawn by the processor will depend on the number of its sections that are active, and the speed at which they operate, all of which can vary dynamically. Similarly, the supply voltage can vary due to external disturbances. Variations in $i_\text{p}$ and $v_\text{s}$ will in turn cause variations in $v_\text{p}$, which are important to understand because the processor will not operate properly if the variations in $v_\text{p}$ are too large.
(2A) (5%) Derive a second-order differential equation that relates $v_p$ to $v_s$ and $i_p$.

$$\text{Diff Eqn: } LC \frac{d^2 v_p}{dt^2} + RC \frac{dv_p}{dt} + v_p = V_s - R i_p - L \frac{di_p}{dt}$$

(2B) (10%) Assume that the supply voltage $v_s$ and the processor current $i_p$ have both been constant for a long time at the values $V_s$ and $I_{p1}$, respectively. Then, at $t = 0$, $i_p$ takes a step such that $i_p(t) = I_{p1} + I_{p2} u(t)$. In response, for $t \geq 0$, $v_p$ takes the form $v_p(t) = V_{p1} + V_{p2} e^{-\alpha t} \cos(\omega t + \phi)$. Determine the constants $V_{p1}$, $V_{p2}$, $\alpha$, $\omega$ and $\phi$ in terms of $R$, $L$, $C$ and the source parameters; you may also express $V_{p1}$, $V_{p2}$, $\alpha$, $\omega$ and $\phi$ in terms of each other as long as only simple back substitution is required to ultimately express them in terms of $R$, $L$, $C$ and the source parameters.

\[ V_{p1} = \frac{V_s - R (I_{p1} + I_{p2})}{1 + R/2L} \]
\[ V_{p2} = \frac{R I_{p2}}{\cos(\phi)} \]
\[ \alpha = \frac{R}{2L} \]
\[ \omega = \sqrt{\frac{1}{LC} - \frac{\alpha^2}{2}} \]
\[ \phi = \tan^{-1}\left( \frac{1}{\omega R C} - \frac{\alpha}{\omega} \right) \]

(2A) \[ v_s = R \left( i_p + C \frac{dv_p}{dt} \right) + L \frac{dv_p}{dt} \left( i_p + C \frac{dv_p}{dt} \right) + v_p \]

(2B) \[ \text{Diff Eqn } \Rightarrow \alpha \text{ and } \omega. \text{ Final Value } \Rightarrow V_{p1}. \]

Initial condition for $v_p$ \[ V_{p2} \cos(\phi) + V_{p1} = V_s - R I_{p1} \]

Initial condition for $\frac{dv_p}{dt} \Rightarrow$
\[ -\alpha V_{p2} \cos(\phi) - \omega V_{p2} \sin(\phi) = -\frac{I_{p2}}{C} \Rightarrow \]
\[ -\alpha R I_{p2} - \omega R I_{p2} \tan(\phi) = -\frac{I_{p2}}{C} \Rightarrow \]
\[ \phi = \tan^{-1}\left( \frac{1}{\omega R C} - \frac{\alpha}{\omega} \right) \]
(2C) Assume that the supply voltage $v_S$ and the processor current $i_P$ have both been constant for a long time at the values $V_S$ and $I_P$, respectively. Then, at $t = 0$, a noise spike occurs in $v_S$ such that $v_S = V_S + \Lambda \delta(t)$, where $\Lambda$ is a constant. Determine the values of $v_P$ and $dv_P/dt$ just after the noise spike in terms of $R$, $L$, $C$ and the source parameters.

$$v_P(0^+) = V_S - R I_P \quad \quad dv_P/dt(0^+) = \frac{\Lambda}{L C}$$

(2D) A cyclic program is written for the processor that results in $i_P = I_P + I_C \cos(\omega t)$, while $v_S = V_S$, where $I_P$, $I_C$ and $V_S$ are all constants. The steady-state response to the cyclic processor current takes the form $v_P = V_P + V_C \cos(\omega t + \phi)$. Determine the constants $V_P$, $V_C$ and $\phi$ in terms of $R$, $L$, $C$, $\omega$ and the source parameters.

$$V_P = V_S - R I_P$$

$$V_C = I_C \sqrt{\frac{R^2 + \omega^2 L^2}{(1 - \omega^2 L C)^2 + \omega^2 R^2 C^2}}$$

$$\phi = \tan^{-1}\left(\frac{\omega L - \omega R C - \frac{\omega^2 R C}{R}}{R}\right)$$

(2C) Impulse falls across the inductor; $V_P$ is continuous and inductor current steps up by $\frac{\Lambda}{L}$.

(2D) $V_P = -I_C \frac{1}{j \omega C (R + j \omega L)} = -I_C \frac{R + j \omega L}{1 - \omega^2 L C + j \omega R C}$

$$= -I_C \frac{R + j \omega L (1 - \omega^2 L C) - j \omega R^2 C}{(1 - \omega^2 L C)^2 + (\omega R C)^2}$$

6
This problem studies the op-amp circuit shown below. Note that its output voltage is the voltage \( v_{\text{OUT}} \). Assume that the op-amp is ideal. Also, assume that the circuit is stable. That is, assume that all voltages are finite for a finite input voltage \( v_{\text{IN}} \).
(2A) Determine the Thevenin-equivalent resistance of the op-amp circuit at its output port as a function of the circuit parameters. Note that the output port is the port at which $v_{OUT}$ is defined.

Let $v_{IN} = 0$ and inject a test current into the positive output terminal.

\[ v_+ = v_- = v_{OUT} \quad \text{... ideal op-amp} \]

\[ v_{AMP} = v_- \frac{R_{1} + R_{2}}{R_{1}} \quad \text{... Voltage divider from } v_{AMP} \text{ to } v_- \]

\[ i_{Test} = \frac{n_{OUT}}{R} + \frac{v_{OUT} - v_{AMP}}{R} \quad \text{... KCL at positive } v_{OUT} \text{ terminal} \]

Substitutions \[ v_{OUT} = \frac{R \cdot R_{1}}{R_{1} - R_{2}} \cdot i_{Test} \]

Thevenin Resistance $R_{TH}$
(2B) Determine the Thevenin-equivalent voltage and the Norton-equivalent current of the op-amp circuit at its output port as a function of the circuit parameters.

\[ V_+ = V_- = V_{out} \quad \cdots \quad \text{Ideal op-amp} \]

\[ V_{Amp} = V_- \frac{R_1 + R_2}{R_1} \quad \cdots \quad \text{Voltage divider from } V_- \text{ to } V_- \]

\[ \frac{V_{out} - V_{in}}{R} + \frac{V_{out} - V_{Amp}}{R} = 0 \quad \cdots \quad \text{KCL at positive } V_{out} \text{ terminal} \]

Substitutions \( \Rightarrow \)

\[ V_{out} = V_{in} \frac{R_1}{R_1 - R_2} \]

\[ \text{Thevenin Voltage } V_{TH} \]

\[ \text{Norton Current} = \frac{V_{TH}}{R_{TH}} = \frac{V_{in}}{R} \]

\[ I_N \]
(2C) The op-amp circuit is connected to a load resistor at its output port as shown below. Assume that \( R_1 = R_2 = R \). Determine the current \( i_L \) into the load as a function of \( v_{IN} \), \( R \) and \( R_L \).

As \( R_1 = R_2 \Rightarrow R \), \( v_{TH} \to \infty \), \( R_{TH} \to \infty \) and \( i_N = \frac{v_{IN}}{R} \), the op-amp circuit becomes a current source.

\[
    i_L = \frac{v_{IN}}{R}
\]
(2D) Assume that the op-amp output voltage $v_{\text{AMP}}$ must satisfy $|v_{\text{AMP}}| < V_S$ for the op-amp to operate properly. Still assuming that the op-amp circuit is connected to a load resistor as shown in Part 2C, and still assuming that $R_1 = R_2 = R$, determine the allowable range of $R_L$ that ensures proper operation of the circuit. Do so in terms of $v_{\text{IN}}$, $R$ and $V_S$.

\[ N_{\text{OUT}} = \frac{v_{\text{IN}}}{R} \cdot R_L \quad \text{... Current source into load resistance} \]

\[ v_{\text{AMP}} = 2 \cdot N_{\text{OUT}} \quad \text{... Voltage divider from } N_{\text{AMP}} \text{ to } V_- \]

\[ \left| 2 \frac{v_{\text{IN}}}{R} R_L \right| < V_S \]

\[ 0 < R_L < \frac{V_S R}{2 |v_{\text{IN}}|} \quad \text{... } R > 0 \text{ and } V_S > 0 \]
(2E) The load resistor in Part 2C is replaced by a load capacitor having capacitance $C$. Assume that $v_{\text{IN}}$ is constant, and that $v_{\text{OUT}}(t = 0) = 0$. Still assuming $R_1 = R_2 = R$, determine $v_{\text{OUT}}(t)$ for $t > 0$ as a function of $v_{\text{IN}}$, $R$ and $C$.

\[ i_{\text{IN}} = C \frac{dN_{\text{OUT}}}{dt} = \frac{v_{\text{IN}}}{R} \]

\[ N_{\text{OUT}} = \frac{1}{RC} \int_{0}^{t} v_{\text{IN}} \, dt = \frac{t}{RC} N_{\text{IN}} \]
(2F) The load resistor in Part 2C is replaced by a load inductor having inductance $L$. Assume that $v_{IN}$ has been zero for all time prior to $t = 0$, and that $v_{IN} = Kt$ thereafter. Still assuming $R_1 = R_2 = R$, determine $v_{OUT}(t)$ for $t > 0$ as a function of $v_{IN}$, $R$, $L$ and the constant $K$.

$$v_{OUT} = L \frac{di}{dt} = \frac{KL}{R}$$
Problem 3 – 25%

The circuit shown below is initially at rest such that $v_C(0^-) = 0$ and $i_L(0^-) = 0$. At $t = 0$, the current source steps up to produce 1 A. The figures shown below plot four waveforms within the circuit following the current step. Their horizontal axes all show time measured in seconds [s]. Their vertical axes show either voltage measured in Volts [V] or current measured in Amperes [A], as appropriate. Note that the initial slopes of the waveforms in Figures 1 and 3 are positive, while the initial slopes of the waveforms in Figures 2 and 4 are zero.
(3A) For each variable listed below, identify the corresponding figure by circling the figure number. Briefly explain your reasoning in the space below.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Figure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_C$</td>
<td>1</td>
</tr>
<tr>
<td>$i_C$</td>
<td>2</td>
</tr>
<tr>
<td>$i_L$</td>
<td>3</td>
</tr>
</tbody>
</table>

At $t=0$, the sourced current goes entirely into the capacitor. Only Figure 2 shows this behavior. At $t=0$ the capacitor voltage therefore increases linearly, and as $t \to \infty$ it settles to $R \cdot 1A$. Only Figure 1 shows this behavior. At $t=0$ the inductor current is proportional to the integral of the capacitor voltage, and hence grows quadratically. It settles to 1A as $t \to \infty$. Only Figure 4 shows this behavior. Figure 3 is the inductor voltage.
(3B)  Estimate the value of \( R \). A numerical answer with appropriate units is expected.

\[ V_c(\infty) = R \cdot 1 \text{A}. \quad \text{From Figure 1,} \]

\[ R = 1 \Omega. \]

(3C)  Estimate the value of \( C \). A numerical answer with appropriate units is expected.

At \( t=0 \), \( C \frac{dV_c}{dt} = 1 \text{A}. \quad \text{From Figure 1,} \]

\[ \frac{dV_c}{dt}(0) = \frac{1V}{0.001 \text{s}} \quad \text{so that} \]

\[ C = 0.001 \text{F}. \]
(3D) Estimate the value of $L$. A numerical answer with appropriate units is expected.

\[ 2\pi \sqrt{LC} \approx \text{Oscillation Period} \approx 0.0126 \text{ s} \]

\[ L = 0.004 \text{ H} \]

(3E) Estimate the $Q$ of the circuit. A numerical answer is expected.

\[ Q = \frac{\sqrt{LC}}{R} = 2 \]
For each statement below, circle the correct completion. *Briefly explain your reasoning in the space below.*

- If the resistance $R$ is increased, the quality factor $Q$ will ...  
  \[ Q = \frac{\sqrt{L/C}}{R} \]
  - decrease.
  - remain unchanged.

- If the capacitance $C$ is increased, the quality factor $Q$ will ...  
  - increase.
  - decrease.
  - remain unchanged.

- If the inductance $L$ is increased, the quality factor $Q$ will ...  
  - increase.
  - decrease.
  - remain unchanged.
After a long time $T$, the current source steps down to turn off. For each statement below concerning the subsequent decay of stored energy, circle the correct completion. Briefly explain your reasoning in the space below.

- If the resistance $R$ is increased, the time at which the stored energy falls to half its value at time $T$ will …
  
  … increase.  \( \quad \)  \( \quad \) decrease.  \( \quad \) remain unchanged.

- If the capacitance $C$ is increased, the time at which the stored energy falls to half its value at time $T$ will …
  
  … increase.  \( \quad \)  \( \quad \) decrease.  \( \quad \) remain unchanged.

- If the inductance $L$ is increased, the time at which the stored energy falls to half its value at time $T$ will …
  
  \( \quad \) increase.  \( \quad \) decrease.  \( \quad \) remain unchanged.

\[ \text{Homogeneous KVL} \rightarrow \ \frac{1}{C} \int i_L \, dt + Ri_L + L \frac{di_L}{dt} = 0 \]

\[ \Rightarrow \ \frac{d^2 i_L}{dt^2} + \frac{R}{L} \frac{di_L}{dt} + \frac{1}{LC} i_L = 0 \]

\[ \uparrow \Rightarrow e = -\frac{R}{2L}t \]
Problem 3: Integrator – 25%

Consider the circuit shown above with switch S1 *initially open* and the source is a constant 5V. Assume that the Op-Amp is ideal.

(3A) 6 pts Find $v_{\text{OUT}}(t)$.

The open switch means the capacitor is disconnected from the circuit and no current can flow into or out of it. Furthermore, the disconnected node can float to any potential, thus the capacitive voltage drop cannot constrain the circuit potential in any way. It thus can be completely ignored in this part of the circuit. The remaining circuit is a straight-forward inverting op-amp with gain $\frac{100\,\Omega}{1\,k\Omega} = 0.1$ thus $v_{\text{OUT}} = -0.1v_{\text{IN}} = -0.5V$. Students commonly lost credit for neglecting to provide units or providing an incorrect sign.

$v_{\text{OUT}}(t) = -0.5V$
(3B) 6 pts Suppose at $t = 0$, switch S1 is moved from an open to a closed position. Assume that the capacitor voltage $v_C(0) = 0$ (i.e. there is no initial charge). Find the time constant $\tau$ of the system after this event, i.e. for $t > 0$.

Because the negative op-amp input acts as a virtual ground, the only impedance seen by the capacitor is the 100 $\Omega$ resistor in parallel with it. Thus $\tau = RC = 100 \Omega \times 100 \text{nF} = 100 \Omega \times 10^{-9} F = 10 \times 10^{-6} \Omega F = 10 \mu s$.

\[ \tau = 10 \mu s \]
(3C) 7 pts Solve for $v_{OUT}(t)$ in the situation described in problem (3B) for $t > 0$. Leave $\tau$ as a symbol, rather than substituting your answer from (3B).

The capacitor starts out uncharged, and we know that in the absence of infinite currents, the capacitor voltage must be continuous, thus with the switch suddenly closed this means that $v_{OUT}(t = 0 + \epsilon) = v_{OUT}(t = 0 - \epsilon) = 0$ V. Meanwhile, at $t = \infty$ we know that steady state must be reached thus the capacitor can be treated as an open circuit and thus we have the same solution as for part A, meaning $v_{OUT} = -0.5$ V. Using the standard formula for 1st-order step response of circuits that approach a steady state, $v(t) = (v_{init} - v_{final})e^{\Delta t/\tau} - v_{final}$, this means that $v_{OUT}(t > 0) = -0.5V + 0.5Ve^{-t/\tau}$.

For $t > 0$, $v_{OUT}(t) = -0.5V + 0.5Ve^{-t/\tau}$
At \( t = \tau \), assume the switch is switched back to being open. Solve for \( v_{\text{OUT}}(t) \) for \( t > \tau \).

As in part A, the capacitor can play no part in the circuit, the circuit output immediately steps to \(-0.5V\)

For \( t > \tau \), \( v_{\text{OUT}}(t) = -0.5V \)
Problem 17: 4 points

\[ v_2(t) \quad v_1(t) \quad v_0(t) \]

\[ \begin{align*}
R & \quad 2R \\
R & & R \\
3R & \quad 2R \\
\end{align*} \]

Figure 16.

Derive an expression for \( v_0(t) \) in terms of \( v_1(t) \) and \( v_2(t) \).

*First, superposition at the \( n^+ \) node:

\[
\begin{align*}
\eta^+ &= \frac{R/2}{R/2 + 2R} \eta_1 + \frac{2/3R}{R + 2/3R} \eta_2 \\
\eta^+ &= \frac{1}{5} \eta_1 + \frac{2}{5} \eta_2 \\
\end{align*}
\]

In addition, \( \eta^- = \frac{2}{5} \eta_0 \Rightarrow \eta_0 = \frac{5}{2} \eta^- \)

since \( \eta^- = \eta^+ \),

\[
\eta_0 = \frac{5}{2} \left( \frac{1}{5} \eta_1 + \frac{2}{5} \eta_2 \right)
\]

\[ v_0(t) = \frac{\eta_1}{2} + \eta_2 \]