Massachusetts Institute of Technology  
Department of Electrical Engineering and Computer Science  

6.002 – Circuits & Electronics  
Fall 2018  

Final Exam  

20 December 2018  

Solutions  

Name: ________________________________  

MIT Kerberos ID: ___________________________  

Recitation Time: 11 12 1  

• There are 25 pages in this exam, including this cover page.  
• Please put your name in the space provided above, and circle the time of your recitation.  
• Please do not remove any pages from this exam.  
• Do your work for each question within the boundaries of that question, or on the back of the preceding page. When finished with each part, clearly write your answer for that part into the corresponding answer box or graph.  
• All numerical answers require proper units.  
• All answers must be justified by supporting math and/or explanations in order to receive full credit.  
• This is a closed-book exam, but calculators and a single two-sided page of notes are allowed.  
• Good luck!  

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</table>
Problem 1: Operational Amplifier Circuits – 10%

Consider the circuit shown below. Assume that the op amps are ideal. Determine $v_{OUT}$ in terms of $v_{IN1}$, $v_{IN2}$ and the resistor values.

\[
\begin{array}{c}
\text{\textbf{\textit{V\text{\textsubscript{OUT}}}}} \\
\rule{0.5\textwidth}{0.4pt} \\
\begin{array}{l}
\frac{R_3 R_5 (R_3 + R_4)}{R_3} V_{IN1} + \frac{R_3}{R_3 + R_4} V_{IN2} \\
- \frac{R_3}{R_3 + R_4} V_{IN1} + \frac{R_3}{R_3 + R_4} V_{IN2} \\
\end{array}
\end{array}
\]

With Superposition: $V_b = V_{OUT} \cdot \frac{R_4}{R_3 + R_4} + V_{IN2} \cdot \frac{R_3}{R_3 + R_4}$.

$V_a = V_b$.

At node C: 
\[
\frac{V_{IN1}}{R_1} + \frac{V_{OUT}}{R_3} + \frac{V_{OUT}}{R_2} = 0.
\]

\[
\begin{array}{l}
\frac{V_{IN1}}{R_1} + \frac{V_{OUT} \cdot \frac{R_3}{R_3 + R_4}}{R_3} + \frac{V_{IN2} \cdot \frac{R_3}{R_3 + R_4}}{R_3 + R_4} + \frac{V_{OUT}}{R_2} = 0.
\end{array}
\]

$V_{OUT} = - \frac{1}{R_2} \cdot \frac{1}{R_3} \cdot \frac{R_4}{R_3 + R_4} = \frac{R_3 R_5 (R_3 + R_4)}{R_1} V_{IN1} + \frac{R_3}{R_3 + R_4} V_{IN2}$.
A core function in our ultrasound board this semester was “mixing” the transmitted 40-kHz signal with the received signal at 40 kHz ± δf, where δf was the relatively small deviation (less than tens of Hz) caused by the sensed-object velocity via the Doppler Effect. The two signals were mixed by multiplying them to generate a new signal with components having distinctly different frequencies. One frequency was 80 kHz ± δf, based on the sum of the original frequencies. The other frequency was ±δf, based on the difference of the two original frequencies. In our ultrasound board we kept only the low-frequency component, ±δf, which contained information exclusively about the sensed-object velocity, by running the mixed signal through a Sallen-Key low pass filter.

This problem considers two modifications to the original mixing and filtering approach. First, instead of using 40 kHz for transmission and mixing, we now use 300 kHz for both transmission and mixing. Second, instead of using a low-pass filter to pass only the low-frequency component we use a notch filter to remove only high-frequencies over a specific narrow band. The notch filter is shown below.

The notch filter should locate its notch at 600 kHz, or 3.77 Mrad/s. Further, let Q = 30 and R = 1 kΩ. Determine L and C.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L</strong></td>
<td></td>
</tr>
<tr>
<td><strong>C</strong></td>
<td></td>
</tr>
</tbody>
</table>
Problem 2

There's a few ways to approach this. Depending on your background and fluency with circuits you can recognize that this is a series RLC circuit. It will resonate at

\[ \omega_o = \frac{1}{\sqrt{LC}} \]

And that means (Because it is in series) that the \( L \) and \( C \) will cancel out (go to zero), resulting in the notch at resonance. Since it is also a series RLC circuit, you may also know (from having previously derived in other settings that the damping coefficient \( \alpha \) of such a system will be

\[ \alpha = \frac{R}{2L} \]

And finally, for any circuit, its “quality”, \( Q \) is based on the expression:

\[ Q = \frac{\omega_o}{2\alpha} \]

With all of this in mind we can first determine a relationship between \( L \) and \( C \) from the desired Notch location...that means we know that:

\[ 3.77 \times 10^6 \text{ rad/s} = \frac{1}{\sqrt{LC}} \]

We also know that we'd like the \( Q \) to be 30. So this means:

\[ Q = 30 = \frac{\omega_o}{2\alpha} = \frac{1}{\sqrt{LC}} \cdot \frac{2R}{2L} = \frac{\sqrt{L}}{C} \]

So we have two equations and three unknowns. Thankfully the resistor is also specified in the problem (1 k\( \Omega \))...so we know:

\[ 30 \times 10^3 = \sqrt{\frac{L}{C}} \]

Multiplying the two equations by one another then means:
(3.77 \times 10^6 \text{ rad/s}) (30 \times 10^3) = \left( \sqrt{\frac{L}{C}} \right) \left( \frac{1}{\sqrt{LC}} \right)

113.1 \times 10^9 1/F = \frac{1}{C}

So:

\[ C = 8.85 \text{ pF} \]

Sweet...now we just let whichever equation we feel like take us the rest of the way home, friend. I feel like picking the second one since it is a Wednesday and that's how I live my life, but you do you.

\[ 30 \times 10^3 = \sqrt{\frac{L}{8.85 \times 10^{-12}}} \]

so...

\[ (900 \times 10^6) \times (8.85 \times 10^{-12}) = L \]

\[ L = 7.956 \text{ mH} \]
Problem 3: Sinusoidal Steady State – 10%

This problem studies the network shown below which is driven in the sinusoidal steady state by the input voltage \( v_1 = V_1 \cos(\omega t) \). Its output voltage is given by \( v_O = V_2 \cos(\omega t + \phi) = \Re \{ \tilde{V}_2 e^{j\omega t} \} \).

\[ \begin{align*}
\tilde{V}_2 &= \frac{R_{Z+} \frac{1}{j\omega C}}{R_{Z+} \frac{1}{j\omega C} + j\omega L + R_1 } \\
&= \frac{\frac{R_2}{1 + j\omega R_2 C}}{1 + j\omega R_2 C} = \frac{\frac{R_2}{R_2 + \frac{1}{j\omega C}}}{\frac{R_2}{1 + j\omega R_2 C} + j\omega L + R_1 } \\
&= \frac{\frac{1}{LC}}{(j\omega)^2 + j\omega \left( \frac{R_1}{L} + \frac{1}{R_2 C} \right) + \frac{R_1 + R_2}{R_2 LC}} \\
\end{align*} \]

(3A) Determine the transfer function \( \tilde{V}_2 / V_1 \).
Determine $\omega_1$ such that the output voltage is given by $v_o = V_2 \sin(\omega_1 t)$ where $V_2$ is both real and positive.

\[
\begin{array}{|c|}
\hline
\omega_1 \\
\hline
\end{array}
\]

\[
(j \omega)^2 = -\frac{R_1 + R_2}{R_2 LC}
\]

\[
\omega = \sqrt{\frac{R_1 + R_2}{R_2} \frac{1}{LC}}
\]
Problem 4: Linear Amplifier Circuits – 15%

This problem concerns the linear amplifier having the equivalent circuit shown below. Note that $\beta$ is a dimensionless gain, and that, except for Part B, $C = 0 \text{ F}$.

(4A) The amplifier is driven by the Thevenin equivalent source comprising $v_S$ and $R_S$, and in turn drives the resistive load $R_L$, as shown below. Determine $v_L$ as a function of $v_S$ and the circuit parameters. Assume that $C = 0 \text{ F}$.

\[
V_{out} = \frac{R_{out} R_L}{R_{out} + R_L} \beta \frac{v_S}{R_S + R_{in}}
\]
An amplifier connection involving feedback is shown below. Develop a differential equation for $v_L$ that can be used to assess its stability, and then use the differential equation to determine the range of $\beta$ for which the connection is stable. To simplify the algebra, assume $R_{IN} \gg R_S$, and $R_{OUT} \gg R_L$, and then make appropriate approximations.

\[ C \frac{dv_L}{dt} + v_L \left[ \frac{1}{R_L} + \frac{1}{R_F + R_S} - \frac{\beta R_S}{(R_S + R_F) R_{IN}} \right] = 0 \]

**Range for $\beta$**

\[ \beta < \frac{R_{IN}}{R_S} \frac{R_S + R_F + R_L}{R_L} \]

Stability is a property of the homogeneous system. So, set $V_S = 0$ and use $R_{OUT} \gg R_L$.

\[ \frac{V_L}{R_L} + C \frac{dv_L}{dt} - \beta i + \frac{V_L}{R_F + R_S} \approx 0 \quad \text{and} \quad i \approx \frac{V_L}{R_S + R_F} \frac{1}{R_{IN}} \]

Substitution $\Rightarrow C \frac{dv_L}{dt} + v_L \left[ \frac{1}{R_L} + \frac{1}{R_F + R_S} - \frac{\beta R_S}{(R_S + R_F) R_{IN}} \right] \approx 0$

Must be positive for stability.
(4C) Consider again the amplifier connection from Part B. Assuming that \( \beta \) is chosen so that the amplifier connection is stable, determine \( v_L \) as a function of \( v_S \) and the circuit parameters. Assume steady-state operation with \( C = 0 \text{ F} \).

\[
\begin{array}{c|c}
\hline
v_L & \hline
\frac{R_L R_F (1 + \beta R_F / R_L)}{(R_L + R_F)(R_S + R_F) - R_L R_S (1 + \beta R_F / R_L)} v_S \\
\hline
\end{array}
\]

Again use \( R_{out} \gg R_L \).

\[
\begin{align*}
K_CL & \Rightarrow \frac{e - v_L}{R_F} + \frac{e - v_S}{R_S} + \frac{e}{R_{IN}} = 0 \Rightarrow e = \frac{v_S R_F}{R_F + R_S} + \frac{v_L R_S}{R_F + R_S} \\
K_CL & \Rightarrow \frac{v_L}{R_L} + \frac{v_L - e}{R_F} - \beta \frac{e}{R_{IN}} = 0 \Rightarrow v_L = \frac{R_L R_F}{R_L + R_F} \left( \frac{1}{R_F} + \frac{\beta}{R_{IN}} \right) e \\
Combined & \Rightarrow v_L = \frac{R_L R_F}{R_L + R_F} \left( \frac{1}{R_F} + \frac{\beta}{R_{IN}} \right) \left( \frac{R_S}{R_F + R_S} v_L + \frac{R_F}{R_F + R_S} v_S \right) \\
\end{align*}
\]

\[
\begin{align*}
v_L & \approx \frac{R_L R_F^2}{R_L + R_F} \left( \frac{1}{R_F} + \frac{\beta}{R_{IN}} \right) \left( \frac{R_S}{R_F + R_S} v_L + \frac{R_F}{R_F + R_S} v_S \right) \\
& \approx \frac{R_L R_F}{R_F + R_S} \left( \frac{1}{R_F} + \frac{\beta}{R_{IN}} \right) v_S \\
\end{align*}
\]
An electronic quartz crystal is an electromechanical resonator actuated through the piezoelectric properties of its mechanical crystal. A circuit model of the electronic device, having electrical terminal variables $i$ and $v$, is shown below. The left-hand side models the piezoelectrically-driven compressive mechanics of the crystal, representing velocity as voltage, and force as current. The compressive mechanics of the crystal comprise a spring with stiffness $K$ modeled as an inductor, a mass with mass $M$ modeled as a capacitor, and a damper with viscous damping $B$ modeled as a resistor. The piezoelectric electromechanical energy conversion process is modeled by two dependent sources with coefficient $A$ having units of [N/V] and [C/m]. The capacitance $C$ is the capacitance between the electrodes attached to the crystal.

![Compressive Mechanics Diagram](image)

(5A) Determine the electrical impedance of the electronic crystal as measured at its electrical terminals where $v$ and $i$ are defined.

<table>
<thead>
<tr>
<th>Electrical Impedance at $v$-$i$ Terminals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{j \omega} \cdot \frac{1}{C + \frac{A^2}{K - M \omega^2 + j \omega B}}$</td>
</tr>
</tbody>
</table>

Mechanics $\Rightarrow \frac{A \ddot{u}}{\ddot{u}} = \frac{K}{j \omega} + j \omega M + B$

Electrics $\Rightarrow \ddot{v} = j \omega C \ddot{u} + A \ddot{u} = j \omega C \ddot{v} + \frac{A^2 v}{\frac{K}{j \omega} + j \omega M + B}$

$Z = \frac{\ddot{v}}{\ddot{u}} = \frac{1}{j \omega C + \frac{A^2}{\frac{K}{j \omega} + j \omega M + B}} = \frac{1/j \omega}{C + \frac{A^2}{K - M \omega^2 + j \omega B}}$
(5B) Assume that the electrical terminals of the electronic crystal are short circuit such that \( v = 0 \). Determine the frequency at which the natural/homogeneous response of the compressive mechanics would respond. Said differently, determine the frequency at which \( u(t) \) would oscillate. To simplify this problem, let \( B = 0 \).

<table>
<thead>
<tr>
<th>Natural Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{\frac{K}{M}} )</td>
</tr>
</tbody>
</table>

\[ v = 0 \Rightarrow Au = 0 \]

Also, \( B = 0 \)

\[ \omega = \sqrt{\frac{1}{\frac{1}{K} \cdot M}} = \sqrt{\frac{K}{M}} \]
Assume that the electrical terminals of the electronic crystal are open circuited such that \( i = 0 \). Determine the frequency at which the natural/homogeneous response of the compressive mechanics would respond. Said differently, determine the frequency at which \( u(t) \) would oscillate. To simplify this problem, again let \( B = 0 \).

\[
\omega = \sqrt{\frac{CK + A^2}{CM}}
\]

Electrics with \( i = 0 \) \( \Rightarrow \) \( Au = -C \frac{dv}{dt} \) \( \Rightarrow \) \( v = -\frac{A}{C} \int u \, dt \)

Reflecting this to the mechanical side,

\[
\begin{align*}
\hfill + & \iff \hfill + \hfill - \\
\hfill u \mbox{ } A \mbox{ } v & \iff \hfill - \mbox{ } \frac{A^2}{C} \int u \, dt \iff \hfill \frac{C}{A^2}
\end{align*}
\]

So the equivalent mechanical system is

\[
\begin{array}{l}
\frac{1}{K} \quad M \quad \frac{C}{A^2} \quad u \\
\hfill u \mbox{ } \iff \hfill \omega = \sqrt{\frac{1}{M \left( \frac{C}{A^2 + \frac{1}{k}} \right)}}
\end{array}
\]

\[
\omega = \sqrt{\frac{CK + A^2}{CM}}
\]
Problem 6: Transistor Amplifiers – 10%

A speaker, which converts an electric signal into an acoustic signal, is modeled here simply as an 8-Ω resistor, as shown below. It is desired to drive this speaker using the output from a microcontroller similar to that which was used in lab. An electrical model of the digital-to-analog converter (DAC) output from the microcontroller is also shown below. Assume $0 \leq v_{\text{SIG}} \leq 5 \text{ V}$.

(6A) Suppose that the DAC is used to drive the speaker directly as shown below. Determine the maximum voltage that can be applied to the speaker, and the maximum instantaneous power that can be delivered to the speaker.

<table>
<thead>
<tr>
<th>Maximum Speaker Voltage</th>
<th>Maximum Speaker Power</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
An alternative means of driving the speaker is to do so through a MOSFET amplifier as shown below. Determine the maximum voltage that can be applied to the MOSFET at $v_{GS}$, and the corresponding gate current $i_G$.

<table>
<thead>
<tr>
<th>Maximum $v_{GS}$</th>
<th>$i_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Following Part B, let the MOSFET be characterized by a threshold voltage of $V_T = 1 \text{ V}$ and a conduction parameter of $K = 0.75 \text{ A/V}^2$. In this case, determine the MOSFET current $i_D$ and the power delivered to the speaker for $v_{GS} = 2.5 \text{ V}$. Assume that $V_{DD}$ is so large that the MOSFET operates in its saturation region such that $i_D = 0.5K(v_{GS} - V_T)^2$.

<table>
<thead>
<tr>
<th>$i_D$</th>
<th>Speaker Power</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Problem 6

6.a If the Thevenin equivalent of the speaker is an 8 Ohm resistor and the Thevening equivalent of the DAC is a voltage source that maxes out at 5V with a 100 Ohm series resistor (its Thevening equivalent resistance), that means when put together they’ll form a Voltage divider of the form:

\[ v_{\text{SPEAKER}} = v_{\text{DAC}} \frac{8 \Omega}{8 \Omega + 100 \Omega} \]

The max voltage is going to happen when the DAC voltage is maxed at 5V (obv), so:

\[ v_{\text{SPEAKER,\text{MAX}}} = 5 \text{ V} \frac{8 \Omega}{8 \Omega + 100 \Omega} = \frac{40 \text{ V}}{108} \]

\[ v_{\text{SPEAKER,\text{MAX}}} \approx 0.37 \text{ V} \]

6.b The MOSFET’s gate looks like an infinite resistor to the DAC (in 6.002 anyways), so that means there will be no current into it

\[ i_G = 0 \text{ A} \]

And because of that, no voltage drop occurs over the DAC’s Thevenin resistance so:

\[ v_{\text{GS,\text{MAX}}} = 5 \text{ V} \]

6.c

With the parameters specified, the current through the MOSFET will be:
Or by doing the numbers out:

\[
i_D = \frac{0.75 \text{ A/V}^2}{2}(2.5 \text{ V} - 1.0 \text{ V})^2 =
\]

\[
i_D = 0.84375 \text{ W}
\]

Because \( p_{\text{Resistor}} = i^2R \), we can then say:

\[
p \approx 5.7 \text{ W}
\]

This is significantly more power than what the DAC could deliver. Thanks, Transistor! You’re the best.
Problem 7: Small-Signal Analysis – 10%

This problem studies the linearized small-signal operation of the receiver in the optical communication link shown below. The communication link comprises a LED-based transmitter and a phototransistor-based receiver. Also shown below is the relation between the phototransistor current $i_C$ and its received light intensity $\lambda_R$ assuming that $v_{OUT} \geq 0.2$ V. Note that a phototransistor is similar to transistors described in class, except that the current through it is controlled by light.

(7A) Let the intensity of the light received by the phototransistor take the form $\lambda_R = \Lambda_R + \lambda_r$ where $\Lambda_R$ is a large-signal bias intensity and $\lambda_r$ is a small-signal variation around that bias. Assume that $\Lambda_R = 0.6$ Lumens, determine $V_{OUT}$, the corresponding large-signal component of $v_{OUT}$.

<table>
<thead>
<tr>
<th>$V_{OUT}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>

$$i_C = I_T \left(e^{\lambda_R/\Lambda_T} - 1\right)$$
$$I_T = 1 \times 10^{-11} \text{ A}$$
$$\Lambda_T = 0.03 \text{ Lumens}$$
Let $v_{OUT} = V_{OUT} + v_{out}$ where $V_{OUT}$ is the large-signal bias output voltage and $v_{out}$ is a small-signal variation around that bias. Develop a small-signal model for the receiver that can be used to determine the small-signal component $v_{out}$ from $\lambda_r$.

<table>
<thead>
<tr>
<th>Small-Signal Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
</tbody>
</table>
(7C) Determine the small-signal receiver gain $v_{out}/\lambda_r$.

<table>
<thead>
<tr>
<th>Small-Signal Gain $v_{out}/\lambda_r$</th>
<th></th>
</tr>
</thead>
</table>
Problem 7

7.a The large signal output voltage is going to be based off of:

\[ V_{OUT} = 12 \text{ V} - I_C \text{ (1 kΩ)} \]

We can figure out \( I_C \) from evaluating the expression for \( i_C \) with only \( \Lambda_R \) applied for \( \lambda_R \). When we do that:

\[ I_C = 1 \times 10^{-11} \left( e^{\frac{0.6}{0.05}} - 1 \right) = 4.85 \text{ mA} \]

That means:

\[ V_{OUT} = 12 \text{ V} - 4.85 \text{ mA} \times 1 \text{ kΩ} \]

\[ V_{OUT} \approx 7.15 \text{ V} \]

7.b

The small signal model comes from taking the derivative of our complete signal expression with respect to the input and analyzing it at the large signal bias point. That means here:

\[ \left. \frac{di_C}{d\lambda_R} \right|_{\lambda_R} = \frac{I_T}{\Lambda_T} e^{\frac{\lambda_R}{\Lambda_T}} \left|_{\lambda_R=\Lambda_R} \right. \]

Which for our numbers of \( \Lambda_R \) means a small signal model of the following, which is a light-controlled current source with small signal gain of:

\[ 0.1617 \text{ A/Lumen} \]

This then could be placed within a circuit of the following form since All DC voltages become 0 (they have no small signal component). Note you could have both drawn or not drawn the light source in an explicit form here. It is fine either way.
Note: $g_m$ is usually representing a transconductance or something like that, but we'll use it for short-hand here since it is often used as our small signal “gain”.

7.c

If we analyze the small signal circuit presented in the previous section we can state that

$$v_{out} = -g_m (1 \text{ k\Omega})$$

If we analyze for our particular value of $\Lambda_R$ we get:

$$v_{out}/\lambda_r = -161.7 \text{ V/Lumen}$$
Problem 8: Network Input-Output Matching – 10%

This problem is concerned with the three networks shown below, each of which has unspecified device parameters. For each part of this problem, the objective is to use graphical input-output data given in that part to identify the corresponding network, and determine the unspecified device parameters. *Note that the data graphs are drawn to an accuracy that is in rough proportion to the scales of the graph axes.*
Determine which network corresponds to the $v_i - v_o$ input-output data shown below. Indicate the identifier of the network (A, B or C) in the answer box, and determine its unspecified device parameters.

<table>
<thead>
<tr>
<th>Network</th>
<th>Unspecified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$C = 50 \text{ nF}$</td>
</tr>
</tbody>
</table>

\[ T = 50 \mu s = RC = 1k \Omega \times C \]
\[ \rightarrow C = \frac{50 \times 10^{-6} \text{ s}}{10^3 \Omega} = 50 \text{ nF} \]
(8B) Determine which network corresponds to the $v_1-v_0$ input-output data shown below. Indicate the identifier of the network (A, B or C) in the answer box, and determine its unspecified device parameters.

![Circuit Diagrams](image)

<table>
<thead>
<tr>
<th>Network</th>
<th>Unspecified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$R = 40 \ \Omega$</td>
</tr>
</tbody>
</table>

**HPF**

$\omega = \frac{9973 \ \text{rad}}{s}$

$T = 6.3 \ \text{ms}$ is at $L$

$f = \frac{1}{T} \Rightarrow \omega = \frac{2\pi}{T} = \frac{6.3}{6.3 \ \text{ms}} = 10^{3} \ \text{rad} \ \frac{s}{s} = \frac{1}{L} = \frac{R}{L}$

$R = \omega L = \left(10^{3} \ \text{rad} \ \frac{s}{s}\right) \left(40 \ \times 10^{-3} \ \text{H}\right) = 40 \ \Omega$
Determine which network corresponds to the \( v_1-v_0 \) input-output data shown below. Circle the identifier of the network, and determine its unspecified device parameters.

<table>
<thead>
<tr>
<th>Network</th>
<th>Unspecified Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>( C = 0.25 , \text{nF} ) ( L = 2.5 , \mu \text{H} )</td>
</tr>
</tbody>
</table>

Top plot is at resonance due to 90° phase shift from \( v_1 \Rightarrow v_0 \)

\[ Q = \frac{R}{L} = \frac{480}{3} = 60 = \frac{R}{\sqrt{LC}} = \frac{6000}{\sqrt{LC}} \Rightarrow \frac{L}{C} = 10^4 \]

Since \( Q \gg 1 \), \( \omega_d = \frac{1}{\sqrt{LC}} \)

From plot, \( T = 0.16 \, \mu \text{s} \Rightarrow \frac{1}{T} = \frac{1}{0.16 \times 10^6 \, \text{Hz}} \approx 2\pi \times 10^6 \, \text{Hz} = \frac{\omega_0}{2\pi} \Rightarrow \omega_0 = (2\pi)^2 \times 10^6 \, \text{rad/s} \]

\[ (2\pi)^2 = \frac{1}{LC} = \frac{1}{10^6 \, \text{C}^2} \Rightarrow C^2 = \frac{1}{(2\pi)^2 \times 10^6} \Rightarrow C = \frac{1}{(2\pi)^2} \times 10^{-8} \, \text{F} = 39 \, \text{pF} \]

\[ C = 0.25 \, \text{nF} \]

\[ L = 2.5 \, \mu \text{H} \]
Problem 9: First-order PWM Circuits – 15%

This problem concerns the circuit shown below, which can be used in pulse-width modulation applications. Note that the circuit contains a comparator and an N-channel MOSFET.
Consider first the comparator circuit inside the dashed box labeled “A”, having \( v_1 \) and \( v_2 \) as its input and output voltages, respectively. Let \( V_{\text{REF}} = 2 \text{ V} \), \( R_3 = 190 \text{ k}\Omega \), and \( R_4 = 10 \text{ k}\Omega \). The positive and negative supply values are \( \pm V_{\text{SS}} \), respectively, where \( V_{\text{SS}} = 5 \text{ V} \). Assuming that \( v_2 = -V_{\text{SS}} \), determine the maximum value of \( v_1 \) that can switch \( v_2 \) from \(-V_{\text{SS}}\) to \( V_{\text{SS}} \)? Similarly, assuming that \( v_2 = V_{\text{SS}} \), determine the minimum value of \( v_1 \) that can switch \( v_2 \) from \( V_{\text{SS}} \) to \(-V_{\text{SS}}\)? Plot and clearly label the input-output relationship between the voltages \( V_1 \) and \( V_2 \) on the axes given below.

<table>
<thead>
<tr>
<th>Maximum ( v_1 ) for ( \uparrow ) Switching</th>
<th>Minimum ( v_1 ) for ( \downarrow ) Switching</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.65V</td>
<td>2.15V</td>
</tr>
</tbody>
</table>

Switching point: \( V_{\text{out}} \cdot \frac{R_4}{R_3+R_4} + V_{\text{ref}} \cdot \frac{R_3}{R_3+R_4} = V_1 \)

For \( -V_{\text{SS}} \rightarrow V_{\text{SS}} \) switching:

\[
V_1 = -5 \text{ V} \cdot \frac{10 \text{ k}\Omega}{200 \text{ k}\Omega} + 2 \text{ V} \cdot \frac{190 \text{ k}\Omega}{200 \text{ k}\Omega}
\]

\[
= 1.9 \text{ V} - 0.25 \text{ V} = 1.65 \text{ V}.
\]

For \( V_{\text{SS}} \rightarrow -V_{\text{SS}} \) switching:

\[
V_1 = 5 \text{ V} \cdot \frac{10 \text{ k}\Omega}{200 \text{ k}\Omega} + 2 \text{ V} \cdot \frac{190 \text{ k}\Omega}{200 \text{ k}\Omega}
\]

\[
= 1.9 \text{ V} + 0.25 \text{ V} = 2.15 \text{ V}.
\]
Now consider the remainder of the circuit, which includes an N-channel MOSFET. A switch model having a threshold voltage of 0 V may be used to describe MOSFET behavior. Thus, the MOSFET is a short for \( v_{GS} \geq 0 \) V and an open for \( v_{GS} < 0 \) V. Finally, the supply voltage is \( V_S = 10 \) V, and \( R_1 = 31 \) k\( \Omega \). In order for the average value of the output voltage \( v_O \) to be 5 V, what must be the value of the resistance \( R_2 \)?

\[
\begin{array}{|c|}
\hline
R_2 \\
\hline
19 \text{ k}\Omega. \\
\hline
\end{array}
\]

Comparator switches at \( V_i = 1.9 \text{ V} \pm 0.25 \text{ V} \), which switches on and off MOSFET and diode for PWM.

\[
\langle V_o \rangle = 5 \text{ V}, \quad \langle V_{R_2} \rangle = \frac{R_2}{R_1+R_2} \times 5 \text{ V} = \langle V_i \rangle = 1.9 \text{ V}.
\]

\[
\frac{R_2}{R_1+R_2} = \frac{1.9}{50} \quad R_1 = 31 \text{ k}\Omega, \quad R_2 = 19 \text{ k}\Omega.
\]
The graph below shows the output voltage $v_O(t)$ after the circuit reaches its cyclic steady state. Using results from Parts (A) and (B), determine the values of $v_A$ and $v_B$. Additionally provide an expression for $v_O(t)$ over the interval $T_1 \leq t \leq T_2$. You may do so in terms of $v_A$, $v_B$, $T_1$, $T_2$, $R_1$, and $L$. An exact expression is expected, however, you may assume that $R_L$ is much smaller than both $R_1$ and $R_2$.

<table>
<thead>
<tr>
<th>$v_A$</th>
<th>$v_B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.66 V</td>
<td>4.34 V</td>
</tr>
</tbody>
</table>

\[
v_A e^{-\frac{t-T_1}{T}} \quad (\tau = \frac{L}{R_L})
\]

Charging/discharging of LC circuit happens at $V = 1.9V \pm 0.25V$,

\[
V_A \cdot \frac{R_2}{R_1+R_2} = 1.9V + 0.25V, \quad V_B \cdot \frac{R_2}{R_1+R_2} = 1.9V - 0.25V,
\]

\[
V_A = 5V + 0.66V = 5.66V, \quad V_B = 5V - 0.66V = 4.34V.
\]

\[
v_O(t) = v_A + \left(0 - v_A\right) \cdot (1 - e^{-\frac{t-T_1}{T}}) = v_A e^{-\frac{t-T_1}{T}} \quad (\text{for} \; T_1 < t < T_2)
\]

\[
\tau = \frac{L}{R_L}
\]