Problem 2: First-Order Networks – 30%

This problem concerns the network shown below. When excited from rest at \( t = 0 \) with a 1-V step the network should produce the output voltage \( v_{\text{OUT}} \) shown below when the output port is unloaded. The expression for the output voltage is

\[
v_{\text{OUT}} = 1 \text{ V} \left( \frac{1}{3} + \frac{2}{3} \left( 1 - e^{-(t/\tau)} \right) \right) u(t)
\]

where the time constant \( \tau \) is 1 ms.
Shown below are eight possibilities for the network. They are labeled A through H. Note that the labels A through F are in the upper right corner of the corresponding network while the labels G and H are in the center. The networks are referenced in Parts A and B.
Consider the eight networks labeled A through H. From these, identify one network that can be used to implement the desired step response given excitation from rest and appropriate values for its resistances and capacitances. Circle the label for that network on the line below, and then clearly explain why the network can provide the desired step response.

Circle: A B C D E F G H

Explain: This problem is most easily answered by considering the system order, and the step response at $t=0$ and $t=\infty$. With $V_{in}=0$ (homogeneous response) all networks are first-order and therefore acceptable. Next examine the response at $t=\infty$ by opening the capacitors and attempting to determine $V_{out}$. For A and H, $V_{out} = V_{in}$, which is acceptable. For C and F, $V_{out} = 0$. For B and D $V_{out}$ is a divided fraction of $V_{in}$. For E and G $V_{out}$ can not be determined this way. Closer inspection shows $V_{out}$ will be determined by a capacitive voltage divider, and will hence be a divided fraction of $V_{in}$. Only A and H are acceptable, so examine them at $t=0$ by shorting the capacitors and attempting to determine $V_{out}$. H provides a voltage division by this analysis, but for A $V_{out}$ can not be determined in this way. Closer inspection shows it too provides a (capacitive) voltage division. So, both A and H are acceptable.
(2B) Analyze the network identified in Part A and determine the values of its two unknown elements. Note that each network has one resistance set to 1 kΩ. Numerical results with appropriate units are expected for the unknown element values.

A: Time constant \( \Rightarrow (C_1 + C_2) \cdot 1 \text{k}\Omega = 1 \text{ms} \)

\[ t = 0 \text{ voltage division} \Rightarrow \frac{C_1}{C_1 + C_2} = \frac{1}{3} \]

Solution \( \Rightarrow C_1 = \frac{1}{3} \mu F \) and \( C_2 = \frac{2}{3} \mu F \)

H: Time constant \( \Rightarrow (R + 1 \text{k}\Omega) C = 1 \text{ms} \)

\[ t = 0 \text{ voltage division} \Rightarrow \frac{1 \text{k}\Omega}{R + 1 \text{k}\Omega} = \frac{1}{3} \]

Solution \( \Rightarrow R = 2 \text{k}\Omega \) and \( C = \frac{1}{3} \mu F \)
Shown below are eight more possibilities for the network. They are labeled I through P. *Note that the labels I through N are in the upper right corner of the corresponding network while the labels O and P are in the center.* The networks are referenced in Parts C and D.
Consider the eight networks labeled I through P. From these, identify one network that can be used to implement the desired step response given excitation from rest and appropriate values for its resistances and inductances. Circle the label for that network on the line below, and then clearly explain why the network can provide the desired step response.

Circle:  I  J  K  L  M  N  O  P

Explain: This problem is most easily answered by considering the system order, and the step response at \( t = 0 \) and \( t = \infty \). With \( V_{IN} = 0 \) (homogeneous response) all networks are first-order and therefore acceptable. Next, examine the response at \( t = \infty \) by shorting the inductors and attempting to determine \( V_{out} \). For J and O, \( V_{out} = V_{IN} \), which is acceptable. For L and M, \( V_{out} = 0 \). For N and P, \( V_{out} \) is a divided fraction of \( V_{IN} \). For I and K \( V_{out} \) cannot be determined in this way. Closer inspection shows \( V_{out} \) will be determined by an inductive voltage divider, and will hence be a divided fraction of \( V_{IN} \). Only J and O are acceptable so examine them at \( t = 0 \) by opening the inductors and attempting to determine \( V_{out} \). J provides a voltage division by this analysis, but for O, \( V_{out} \) can not be determined in this way. Closer inspection shows it to provide a (inductive) voltage division. So, both J and O are acceptable.
(2D) Analyze the network identified in Part C and determine the values of its two unknown elements. Note that each network has one resistance set to 1 kΩ. Numerical results with appropriate units are expected for the unknown element values.

\[ J: \text{Time constant } \Rightarrow \frac{L}{R \cdot (1 \, \text{k} \Omega)} = 1 \, \text{ms} \]

\[ t = 0 \text{ voltage division } \Rightarrow \frac{R}{R + 1 \, \text{k} \Omega} = \frac{1}{3} \]

Solution \( R = \frac{1}{2} \, \text{k} \Omega \) and \( L = \frac{1}{3} \, \text{H} \)

\[ O: \text{Time constant } \Rightarrow \frac{L_1 + L_2}{1 \, \text{k} \Omega} = 1 \, \text{ms} \]

\[ t = 0 \text{ voltage division } \Rightarrow \frac{L_1}{L_1 + L_2} = \frac{1}{3} \]

Solution \( L_1 = \frac{1}{3} \, \text{H} \) and \( L_2 = \frac{2}{3} \, \text{H} \)