MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF ELECTRICAL ENGINEERING AND COMPUTER SCIENCE

FINAL - 6.002 Circuits and Electronics

May 23, 2019

Total Points: 150

Time Limit: 180 minutes

YOUR NAME__________________________________________________________

RECITATION: 11 am 12 pm 1 pm

General Instructions:

1. Please do all of your work in the spaces provided in this examination booklet. Place your answer for each question in the space provided in this booklet.

2. The exam consists of 7 problems on pages 2-24. Please make sure you have all of the pages. Use the space immediately following each question to show your work and the answer to the question. If you need more space, use the back of the preceding page.

3. All sketches must be adequately labeled.

4. You will be graded on both your solution (that is, the work shown) and your final answer. It is possible to get the right answer, but not receive full credit if your reasoning is unclear. A few words of explanation are required.

5. Indicate units for all numerical answers.

6. The exam is closed book, but calculators and two two-sided sheets of notes are allowed.

7. Please do not remove any pages from this exam booklet.

Grade: Problem 1: ( /30) Problem 2: ( /15) Problem 3: ( /15) Problem 4: ( /25) Problem 5: ( /15) Problem 6: ( /30) Problem 7: ( /20)

Total grade:
Problem 1 [30 points]: Second-Order Filter

The circuit shown below is a second-order filter and its op-amp is ideal.

\[ v_{\text{IN}} \]
\[ R_1 \]
\[ C_1 \]
\[ R_2 \]
\[ C_2 \]
\[ R_3 \]
\[ R_4 \]
\[ v'_{\text{OUT}} \]

\[ v_{\text{IN}} \]
\[ R_2 \]
\[ C_2 \]
\[ R_3 \]
\[ R_4 \]

\[ \hat{V}_{\text{out}} \]

a) [10 points] Assume that the filter operates in the sinusoidal steady state with \( v_{\text{IN}} = \Re\{\hat{V}_{\text{in}}e^{j\omega t}\} \) and \( v_{\text{OUT}} = \Re\{\hat{V}_{\text{out}}e^{j\omega t}\} \) where \( \hat{V}_{\text{in}} \) and \( \hat{V}_{\text{out}} \) are complex amplitudes. First find a set of two equations at nodes 1 and 2 which, if solved, would yield the input-output transfer function \( H(j\omega) = \hat{V}_{\text{out}}/\hat{V}_{\text{in}}. \) Then, based on these equations find an expression for \( H(j\omega). \)

<table>
<thead>
<tr>
<th>Equation 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ (V_{\text{IN}} - V_1) j\omega C_1 + \frac{(V_{\text{OUT}} - V_1)}{R_1} + (V_2 - V_1) j\omega C_2 = 0 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Equation 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ (V_2 - V_1) j\omega C_2 + \frac{V_2}{R_2} = 0 ]</td>
</tr>
</tbody>
</table>

\[ H(j\omega) = \frac{\hat{V}_{\text{out}}}{\hat{V}_{\text{in}}} \]
\[ \hat{V}_{\text{OUT}} = \frac{j\omega C_1}{-\frac{1}{R_1} + \frac{R_3}{R_3 + R_4} (1 + \frac{j\omega C_2}{R_2}) (j\omega C_1 + \frac{1}{R_1} + j\omega C_2) - \frac{R_3}{R_3 + R_4} j\omega C_2} \]

(More details on next page)
Problem 1

\[ (V_{in} - V_1) \frac{j \omega C_1}{R_1} + \frac{V_{out} - V_1}{R_1} + (V_2 - V_1) \frac{j \omega C_2}{R_2} = 0 \]  \hspace{1cm} (1)

Node 1

\[ (V_2 - V_1) j \omega C_2 + V_2 \frac{R_2}{R_2} = 0 \]
\[ V_2 j \omega C_2 + V_2 \frac{R_2}{R_2} - V_1 j \omega C_2 = 0 \]
\[ V_2 (j \omega C_2 + \frac{1}{R_2}) = V_1 j \omega C_2 \]
\[ V_1 = V_2 (1 + \frac{1}{j \omega C_2 R_2}) \]
\[ V_1 = V_{out} \left( \frac{R_3}{R_3 + R_4} \right) (1 + \frac{1}{j \omega C_2 R_2}) \]  \hspace{1cm} (2)

Node 2

\[ V_{in} j \omega C_1 = V_{out} j \omega C_1 + V_{in}(j \omega C_1 + \frac{1}{R_1}) + V_{in}(1 + \frac{1}{j \omega C_1 R_1}) \]
\[ V_{in} j \omega C_1 = V_{out} j \omega C_1 + V_{in}(j \omega C_1 + \frac{1}{R_1}) + V_{in}(1 + \frac{1}{j \omega C_1 R_1}) \]
\[ V_{in} j \omega C_1 = V_{out} \left[ 1 + \frac{1}{j \omega C_1 R_1} \right] (j \omega C_1 + \frac{1}{R_1}) \]
\[ V_{in} j \omega C_1 = V_{out} \left[ 1 + \frac{1}{j \omega C_1 R_1} \right] (j \omega C_1 + \frac{1}{R_1}) \]
\[ V_{in} j \omega C_1 = V_{out} \left[ 1 + \frac{1}{j \omega C_1 R_1} \right] (j \omega C_1 + \frac{1}{R_1}) \]
\[ V_{in} j \omega C_1 = V_{out} \left[ 1 + \frac{1}{j \omega C_1 R_1} \right] (j \omega C_1 + \frac{1}{R_1}) \]
\[ V_{in} j \omega C_1 = V_{out} \left[ 1 + \frac{1}{j \omega C_1 R_1} \right] (j \omega C_1 + \frac{1}{R_1}) \]

Substitute (2) and (3) in (1).

\[ V_{in} j \omega C_1 - V_{in} j \omega C_1 + \frac{V_{in}}{R_1} - V_1 j \omega C_2 + V_2 j \omega C_2 - V_1 j \omega C_2 = 0 \]
\[ V_{in} j \omega C_1 + \frac{V_{in}}{R_1} - V_1 (j \omega C_1 + \frac{1}{R_1} + j \omega C_2) + V_2 j \omega C_2 = 0 \]
\[ V_{in} j \omega C_1 + \frac{V_{in}}{R_1} - V_1 \left( \frac{R_3}{R_3 + R_4} \right) (j \omega C_1 + \frac{1}{R_1} + j \omega C_2) + V_2 \left( \frac{R_3}{R_3 + R_4} \right) j \omega C_2 = 0 \]
\[ V_{in} j \omega C_1 + \frac{V_{in}}{R_1} - V_1 (j \omega C_1 + \frac{1}{R_1} + j \omega C_2) + V_2 \left( \frac{R_3}{R_3 + R_4} \right) j \omega C_2 = 0 \]

\[ V_{out} = \frac{j \omega C_1}{R_1 \left( 1 + \frac{1}{j \omega C_1 R_1} \right) (j \omega C_1 + \frac{1}{R_1} + j \omega C_2) - \frac{R_3}{R_3 + R_4}} \]
\[ = \frac{R_3 + R_4}{R_3} \left( \frac{j \omega^2}{c_1^2} \right) \]
\[ H(j \omega) = \frac{V_{out}}{V_{in}} = \frac{R_3 + R_4}{R_3} \left( \frac{j \omega^2}{c_1^2} \right) + \frac{j \omega}{c_1 R_1} + \frac{j \omega}{c_1 R_1} \cdot \frac{R_3}{R_3 + R_4} + \frac{j \omega}{c_1 R_1} + \frac{1}{c_1 R_1} - \left( \frac{j \omega}{c_1 R_1} \right)^2 \]
b) [10 points] Assuming that the transfer function is as given by the expression below, find the magnitude and phase of $H(j\omega)$.

$$H(j\omega) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{(j\omega C_1)(j\omega C_2)}{\frac{1}{R_1 R_2} + j\omega(C_1 + C_2) \frac{1}{R_2} + (j\omega)^2 C_1 C_2}$$

<table>
<thead>
<tr>
<th></th>
<th>Magnitude</th>
<th>Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$</td>
<td>H(j\omega)</td>
</tr>
</tbody>
</table>
c) [5 points] Find the low-frequency and high-frequency asymptotes of the transfer function given in Part b). What type of filter is being implemented here?

\[ H(j\omega) = \frac{\ddot{V}_{\text{out}}}{\ddot{V}_{\text{in}}} = \frac{(j\omega C_1)(j\omega C_2)}{\frac{1}{R_1 R_2} + j\omega(C_1 + C_2) \frac{1}{R_2} + (j\omega)^2 C_1 C_2} \]

<table>
<thead>
<tr>
<th>Description</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low-Frequency Asymptote</strong></td>
<td>For ( \omega &lt; \omega_0 ), ( \omega_0^2 + j\omega\left(\frac{1}{C_2 R_2} + \frac{1}{C_1 R_2}\right) + (j\omega)^2 \approx \omega_0^2 )</td>
</tr>
<tr>
<td></td>
<td>( H(j\omega) \approx \frac{(j\omega)^2}{\omega_0^2} = \left(\frac{j\omega}{\omega_0}\right)^2 )</td>
</tr>
<tr>
<td></td>
<td>where ( \omega_0^2 \approx \frac{1}{R_1 R_2 C_1 C_2} )</td>
</tr>
<tr>
<td><strong>High-Frequency Asymptote</strong></td>
<td>For ( \omega &gt; \omega_0 ), ( \omega_0^2 + j\omega\left(\frac{1}{C_2 R_2} + \frac{1}{C_1 R_2}\right) + (j\omega)^2 \approx (j\omega)^2 )</td>
</tr>
<tr>
<td></td>
<td>( H(j\omega) \approx \frac{(j\omega)^2}{(j\omega)^2} = 1 )</td>
</tr>
<tr>
<td><strong>Type of Filter</strong></td>
<td>High-pass filter</td>
</tr>
</tbody>
</table>
d) [5 points] Using the transfer function given in Part b), and given \( C_1 = C_2 = 100 \text{nF} \), design the filter such that it has a quality factor \( Q \) of 4. To do this, determine suitable values of \( R_1 \) and \( R_2 \).

\[
H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{(j\omega C_1)(j\omega C_2)}{\frac{1}{R_1 R_2} + j\omega(C_1 + C_2) \frac{1}{R_2} + (j\omega)^2 C_1 C_2}
\]

| \( R_1 \) and \( R_2 \) | \( R_2 = 64 k\Omega \) \\
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>if ( R_1 = 1 k\Omega )</td>
<td></td>
</tr>
<tr>
<td>( R_2 = 64 k\Omega )</td>
<td></td>
</tr>
</tbody>
</table>

\[
Q = \frac{\omega_0}{\omega_a} = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}} = \frac{C_1 C_2 R_2}{(C_1 + C_2) \sqrt{R_1 R_2 C_1 C_2}}
\]

If \( C_1 = C_2 = C = 100 \text{nF} \)

\[
Q = \frac{C^2 R_2}{\partial C \sqrt{R_1 R_2 C^2}} = \frac{C^2 R_2}{\partial C^2 \sqrt{R_1 R_2}} = \frac{1}{2} \sqrt{\frac{R_2}{R_1}}
\]

To get \( Q = 4 \)

\[
Q = \frac{1}{2} \sqrt{\frac{R_2}{R_1}} = 4 \Rightarrow \sqrt{\frac{R_2}{R_1}} = 8 \Rightarrow \frac{R_2}{R_1} = 64
\]
Problem 2 [15 points]: Power

This problem is concerned with determining the value of a load that maximizes the power delivered to that load.

a) [8 points] A Norton equivalent having current $I$ and resistance $R$ is separately loaded with a voltage source, a current source, and a resistor as shown below. For each load case, determine the value of the load ($V_L$, $I_L$, or $R_L$) in terms of $I$ and $R$ that maximizes the power delivered to the load. Additionally, determine the maximized power delivered to the load.

\[
\begin{array}{llll}
\text{Voltage Load} & V_L & I_L & R_L \\
\text{Current Load} & & & \\
\text{Resistor Load} & & & \\
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
\text{Load Value} & V_L & I_L & R_L \\
\hline
\text{Maximized Power} & & & \\
\hline
\end{array}
\]
b) [7 points] The network shown below contains a resistor having an unknown resistance $R$. Determine the numerical value of $R$ that maximizes the power dissipated in that resistor. Additionally, determine the maximized power.

![Network Diagram](image)

<table>
<thead>
<tr>
<th>$R$</th>
<th>Power</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Problem 2: Power

PART A:

For $V_L$ load:

Let $i$ be the current that flows into the voltage load $V_L$. The voltage drop across $R$ is just $V_L$, and the power delivered to the $V_L$ load, $P$, is

$$ P = iV_L = \left( I - \frac{V_L}{R} \right) \cdot V_L = IV_L - \frac{V_L^2}{R} $$

To find the value of $V_L$ that maximizes the power delivered to the load $V_L$, we differentiate $P$ with respect to $V_L$, set the derivative equal to zero and solve for $V_L$:

$$ \frac{dP}{dV_L} = I - \frac{2V_L}{R} = 0 $$

$$ \Rightarrow V_L = \frac{IR}{2} $$

Since $V_L = (IR)/2$ maximizes the power delivered to the load $V_L$, we substitute this value into the equation for $P$ above to find the maximum power delivered to the load:

$$ P_{MAX} = I\left( \frac{IR}{2} \right) - \frac{(\frac{IR}{2})^2}{R} = \frac{I^2R}{4} $$

For $I_L$ load:

The current that flows into $R$ is just $I - I_L$ and by Ohm’s Law the voltage drop across $R$ is $v = (I - I_L)R$. The voltage drop across the current source load, $I_L$, is the same as that across $R$, and is equal to $v = (I - I_L)R$. Then, the power delivered to the current source load, $P$, is

$$ P = vI_L = (I - I_L)R \cdot I_L = RII_L - R^2I_L^2 $$

To find the value of $I_L$ that maximizes the power delivered to the load $I_L$, we differentiate $P$ with respect to $I_L$, set the derivative equal to zero and solve for $I_L$:

$$ \frac{dP}{dI_L} = RI - 2RI_L = 0 $$

$$ \Rightarrow I_L = \frac{I}{2} $$

Since $I_L = I/2$ maximizes the power delivered to the load $I_L$, we substitute this value into the equation for $P$ above to find the maximum power delivered to the load:

$$ P_{MAX} = RI\left( \frac{1}{2} \right) - R\left( \frac{1}{2} \right)^2 = \frac{I^2R}{4} $$
For $R_L$ load:
The current through the resistor load can be found using current divider and is $i = \frac{R}{R_L + R} \times I$. Then, the power delivered to the $R_L$ load, $P$, is

$$P = i^2 \times R_L = \left( \frac{RI}{R_L + R} \right)^2 R_L = \frac{(IR)^2 R_L}{(R + R_L)^2}$$

To find the value of $R_L$ that maximizes the power delivered to the load $R_L$, we differentiate $P$ with respect to $R_L$, set the derivative equal to zero and solve for $R_L$:

$$\frac{dP}{dR_L} = (IR)^2(R + R_L)^{-2} - 2R_L(IR)^2(R + R_L)^{-3} = 0$$

$$\Rightarrow (IR)^2(R + R_L)^{-2}[1 - 2R_L(R + R_L)^{-1}] = 0$$

$$\Rightarrow \frac{2R_L}{R + R_L} = 1$$

$$\Rightarrow R_L = R$$

Since $R_L = R$ maximizes the power delivered to the load $R_L$, we substitute this value into the equation for $P$ above to find the maximum power delivered to the load:

$$P_{MAX} = \frac{(IR)^2 R}{(R + R)^2} = \frac{I^2 R}{4}$$

***** PART B:
From part A, we learned that the load that maximizes the power delivered to it is a matched load. Specifically, we want $R = R_{th}$. Turning off all the independent sources and looking into the port of $R$, we see the 3 kΩ and the 6 kΩ resistor in parallel, and then the 1 kΩ in series with the parallel resistance. This is $3k\Omega || 6k\Omega + 1k\Omega = 3k\Omega$. So we want $R = 3$ kΩ.

We will next find the Thevenin source value. Suppressing the 1 mA source, we see the 6 V source contributes 4 V across the load. This is because the 3 kΩ, and the 6 kΩ resistors form a voltage divider with a ratio of 2/3. Suppressing the 6 V source, we see -1/3 mA through the 6 kΩ resistor. This is because the 3 kΩ and the 6 kΩ resistors form a current divider of 1/3 for the 6 kΩ branch. This means the current source contributes -2 V to the thevenin source. So, in total, we have a Thevenin source of 2 V across the load.

The Norton equivalent is a 2/3 mA Norton source with a parallel $R_{th}/3k\Omega$

Finally, from the max power formulas we derived in part A, or the one we derived in midterm 1, we see that $P_{MAX} = \frac{I^2 R}{4} = (2/3mA)^2 \times 3k\Omega/4 = 0.33mW$. 
Problem 3 [15 points]: Network identification

The network shown below comprises one independent voltage source, one dependent current source, and two resistors. All but the dependent current source have unknown values. The network also contains a port. A graphical description of the i-v relation measured at the port is also shown below. Finally, it is known that $R_1$ dissipates 2 mW when $v = 0$. Using the information given, determine $V_1$, $R_1$, and $R_2$. 

\[
\begin{array}{|c|c|c|}
\hline
R_1 & R_2 & V_1 \\
\hline
\end{array}
\]
Problem 3: Network identification

From the plot we know that when $i = 0$, $v = 4$ V. $i = 0$ also implies that the dependent source $i/2 = 0$ as well, and can be replaced with an open circuit. As a result, there is no current flowing through the resistors $R_1$ and $R_2$, and there is no voltage dropped across them when $i = 0$. This implies that the voltage $V_1$ appears at the port when $i = 0$. Therefore, $V_1 = v = 4$ V.

When $v = 0$, the port can be replaced with a short and $i = -2$ mA from the plot. Let $i_{R1}$ be the current flowing from the voltage source $V_1$ through $R_1$ in this case. Then, by KCL at the node connecting $R_1$, $R_2$, and the dependent current source $i_2$, we get $i_{R1} + i - \frac{i}{2} = 0$. Substituting $i = -2$ mA into this equation yields $i_{R1} = 1$ mA. In this case, it is given that $R_1$ dissipates 2 mW of power. Therefore, $P = i^2_{R1}R_1 = 0.002$ W $\Rightarrow (0.001 \text{ A})^2R_1 = 0.002$ W $\Rightarrow R_1 = 2000$ $\Omega$.

Also, performing KVL around the loop containing $V_1$, $R_1$, and $R_2$, when $v = 0$, yields $V_1 - R_1i_{R1} + R_2i = 0$. When $v = 0$, we know $i = -2$ mA and $i_{R1} = 1$ mA. Also, substituting $V_1 = 4$ V and $R_1 = 2000$ $\Omega$, we get $4 - 0.001 \times 2000 + R_2(-0.002) = 0 \Rightarrow R_2 = 1000$ $\Omega$. 
Problem 4 [25 points]: Cascode amplifier

The circuit shown below is called a "cascode amplifier". In this problem you will study how it works.

For this problem assume that transistors M1 and M2 both operate in saturation, and that they are characterized by their $K$-factor and threshold voltages, $K_1$ and $V_{T1}$ for M1, and $K_2$ and $V_{T2}$ for M2. $V_{DD}$ is the DC power supply voltage, while $v_{in}$ is the input voltage, and $v_{out}$ is the output voltage. Let $v_{in} = v_{iN} + v_{in}$, where $v_{iN}$ is the large-signal component of the input and $v_{in}$ is the small-signal component of the input. Similarly, let $v_{out} = v_{out} + v_{out}$, where $v_{out}$ is the large-signal component of the output and $v_{out}$ is the small-signal component of the output. Also, note that $V_{T1} < 0$ and $V_{T2} < 0$. For your convenience, we have reproduced below the equations that describe the behavior of a MOSFET.

\[- Cut-off: \ v_{GS} \leq V_T \]
\[i_D = 0\]

\[- Linear \ or \ triode: \ v_{GS} > V_T, \ v_{DS} \leq v_{GS} - V_T \]
\[i_D = K(v_{GS} - V_T - \frac{v_{DS}}{2})v_{DS}\]

\[- Saturation: \ v_{GS} > V_T, \ v_{DS} \geq v_{GS} - V_T \]
\[i_D = \frac{K}{2}(v_{GS} - V_T)^2\]

a) [5 points] Determine the voltage $v_{out}$ as a function of $v_{in}$ and the values of the different circuit elements ($K_1$, $V_{T1}$, $K_2$, $V_{T2}$, $V_{DD}$, $R_D$).

\[
\begin{align*}
v_{out} &= V_{DD} - \frac{R_D K_2}{2} (v_{in} - V_{T2})^2 \\
i_{DSM2} &= \frac{K_2}{2} (v_{in} - V_{T2})^2 \\
v_{out} &= V_{DD} - \frac{R_D K_2}{2} (v_{in} - V_{T2})^2
\end{align*}
\]
b) [5 points] Determine the voltage $V_2$ as a function of the input voltage $v_{in}$ and the values of the different circuit elements ($K_3$, $V_{T3}$, $K_2$, $V_{T2}$, $V_{DD}$, $R_0$).

\[ V_2 = -V_{T_1} + \sqrt{\frac{K_1}{K_f}} (V_{in} - V_{T_2}) \]

\[ i_{D,T_1} = \frac{K_f}{2} (V_{GS,T_1} - V_{T_1})^2 = \frac{K_f}{2} (-V_2 - V_{T_1})^2 = i_{D,T_2} = \frac{K_2}{2} (V_{in} - V_{T_2})^2 \]

\[ (-V_2 - V_{T_1})^2 = \frac{K_2}{K_f} (V_{in} - V_{T_2})^2 \]

\[-V_2 - V_{T_1} = \pm \sqrt{\frac{K_2}{K_f}} (V_{in} - V_{T_2}) \]

\[ V_2 = -V_{T_1} \pm \sqrt{\frac{K_2}{K_f}} (V_{in} - V_{T_2}) \]

*The correct sign is chosen based on the voltage range for transistor saturation.*
c) [5 points] Over what range of \( V_{in} \) will M1 be in saturation? For this part only, you may assume that \( K_1 = K_2 = 1 \text{ mA/V}^2 \), \( V_{T1} = V_{T2} = -2 \text{ V} \), \( V_{DD} = 5 \text{ V} \), and \( R_0 = 1 \text{ k}\Omega \).

<table>
<thead>
<tr>
<th>Range of ( V_{in} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-2 V \leq V_{in} \leq \sqrt{6} - 2 \approx 0.45 V)</td>
</tr>
</tbody>
</table>

**Lower Bound**

For M1 to be in saturation, M2 should flow some current:

\[
V_{in} > V_{T1} = -2V.
\]

**Upper Bound**

\[
\begin{align*}
V_{DS, M1} & > V_{GS, M1} - V_{T1} = -V_2 - V_{T1} \\
V_{out} - V_2 & > -V_2 - V_{T1} \quad \rightarrow \quad V_{out} > 2V.
\end{align*}
\]

\[
\begin{align*}
V_{DD} - \frac{R_0 K_2}{2} (V_{in} - V_{T1})^2 & > 2 \\
5 - \frac{1}{2} (V_{in} + 2)^2 & > 2 \\
\frac{1}{2} (V_{in} + 2)^2 & < 3 \quad \rightarrow \quad V_{in} + 2 \leq \sqrt{6}.
\end{align*}
\]

\[
V_{in} \leq \sqrt{6} - 2 \approx 0.45 V.
\]
d) [5 points] Draw the small-signal equivalent circuit model for the cascode amplifier shown above, and clearly label its circuit elements. For this part only, the element labels may involve input \( V_{\text{in}} \), and MOSFET \( V_{GS}, V_{DS} \) large-signal voltages.

\[
\begin{align*}
g_{m_1} &= \frac{\partial i_D}{\partial V_{GS}} = K_r (-V_T - V_{T1}) \\
where: V_T &= V_{T1} + \sqrt{\frac{K_r}{K_r}} (V_{\text{in}} - V_{T1}) \\
g_{m_2} &= K_r (V_{GS2} - V_{T2}) = K_r (V_{\text{in}} - V_{T2})
\end{align*}
\]
e) [5 points] Calculate the small signal gain $\frac{V_{out}}{V_{in}}$ of the cascode amplifier. For this part only, you may assume that $K_1 = K_2 = 1 \text{ mA/V}^2$, $V_{T1} = V_{T2} = -2 \text{ V}$, $V_{DD} = 5 \text{ V}$, $R_D = 1 \text{ k}\Omega$, $V_{IN} = 1 \text{ V}$.

\[
\begin{array}{c|c}
\text{Small-signal gain} & \\
\hline
A_v = \frac{V_{out}}{V_{in}} = -R_D \cdot g_m = -1 \text{ k}\Omega \cdot 1 \text{ mA/V} \cdot (1V + 2V) = \\
& = -3 \\
\end{array}
\]

The solution can also be obtained by differentiating $V_{out}$ in $A_v$:

\[
A_v = \frac{\partial V_{out}}{\partial V_{in}} = -R_D K_r (V_{IN} - V_T)
\]
Problem 5 [15 points]: Op-Amp with a Nonlinear Element

Consider the circuit below consisting of an ideal op-amp and a nonlinear element. The current-voltage characteristic of the nonlinear element is described by the set of equations shown below and schematically represented in the accompanying graph.

\[ i_N = 2 \times 10^{-3} \left( e^{\frac{v_N}{10^{-3}}} - 1 \right) \quad \text{for} \quad v_N > 0 \]
\[ i_N = 0 \quad \text{for} \quad -3 < v_N < 0 \]
\[ i_N = 1 \times 10^4 v_N + 3 \times 10^4 \quad \text{for} \quad v_N < -3 \]

a) [5 points] Find \( V_O \), the large-signal component of \( v_O \), assuming \( v_A = V_A = 10 \text{ V} \), where \( V_A \) is the large-signal component of \( v_A \).

<table>
<thead>
<tr>
<th>( V_O )</th>
<th></th>
</tr>
</thead>
</table>
b) [10 points] Let $v_a$ be the small-signal component of the input $v_A$. Assuming that the amplitude of $v_a$ is 0.1 V, and that $V_A$ remains at 10 V, determine the amplitude of $v_o$, the resulting small-signal component of the output voltage.

<table>
<thead>
<tr>
<th>Amplitude of $v_o$</th>
<th></th>
</tr>
</thead>
</table>
Problem 5: Op-Amp with a Nonlinear Element

Part (5a):

Since the op-amp is ideal with net negative feedback, \( v_+ = v_- = 0 \) V. Since \( v_A = V_A = 10 \) V, the large signal current flowing from \( v_A \) to \( v_- \) through \( R \) is \( I_R = \frac{v_A - v_-}{R} = \frac{10 \text{ V} - 0}{10 \text{ k}\Omega} \Rightarrow I_R = 1 \text{ mA} \). Since the op-amp is ideal, none of \( I_R \) flows into the negative input terminal of the op-amp and therefore, all of it flows through the nonlinear element in a direction opposite to \( i_N \) or (its large-signal \( I_N \)). Therefore \( i_N = I_N = -I_R = -1 \text{ mA} \).

Since \( i_N = I_N = -1 \text{ mA} \) is strictly negative, the nonlinear element is forced to operate in its linear region where \( v_N < -3 \) V and \( i_N < 0 \). Also, since \( v_O \) is \( v_N \) above \( v_- = v_+ = 0 \) V, we get \( v_O = v_N \) (or \( V_O = V_N \) for large-signal). Now, substituting \( i_N = I_N = -1 \text{ mA} \) into the given \( i-v \) relation for the nonlinear element for negative \( i_N \), we get

\[
I_N = -1 \text{ mA} = 10^4 V_N + 3 \times 10^4 \Rightarrow -10^{-3} = 10^4 (V_N + 3) \Rightarrow V_O = V_N = -3 - 10^{-7} \text{ V} = -3.000001 \text{ V} \approx -3 \text{ V}
\]

[Note: Since units were not specified for \( i_N \) in the given \( i_N - v_N \) equations, \( V_O = -3 - 10^{-4} \text{ V} \), obtained by assuming \( i_N \) as having units of mA in the given \( i_N - v_N \) equations, was also accepted as a correct answer.]

Part (5b):

In small-signal analysis, upon linearization, \( \frac{v_o}{v_a} \approx \left. \frac{dv_O}{dv_A} \right|_{V_O, V_A = 10 \text{ V}} \Rightarrow v_o = \left. \frac{dv_O}{dv_A} \right|_{V_O, V_A = 10 \text{ V}} v_a \)

\[
\Rightarrow v_o = \left. \frac{dv_O}{dv_N} \right|_{v_O, v_A = 10 \text{ V}} \times \left. \frac{di_N}{dv_A} \right|_{v_O, v_A = 10 \text{ V}} v_a
\]

Since \( v_O = v_N \),

\[
\Rightarrow v_o = \left. \frac{dv_N}{di_N} \right|_{v_O, v_A = 10 \text{ V}} \times \left. \frac{di_N}{dv_A} \right|_{v_O, v_A = 10 \text{ V}} v_a
\]

Since \( i_N = 10^4 v_N + 3 \times 10^4 \) when \( V_A = 10 \) V (from Part (a)), we get

\[
\frac{di_N}{dv_N} = 10^4 \Rightarrow \frac{dv_N}{di_N} = 10^{-4} \text{ [V/A]}
\]

Furthermore, we know \( v_- - i_NR = v_A \Rightarrow -i_NR = v_A \Rightarrow \frac{dv_A}{di_N} = -R \Rightarrow \frac{dv_N}{dv_A} = -\frac{1}{R} = -\frac{1}{10000} = -10^{-4} \text{ [A/V]}.
\]
Substituting these values of $\frac{dv_N}{di_N}$ and $\frac{di_N}{dv_A}$, and $v_a = 0.1 \text{ V}$, into the expression above for small-signal $v_o$, we get:

$$v_o = \frac{dv_N}{di_N} \times \frac{di_N}{dv_A} \bigg|_{v_o=v_o, V_A=10\text{ V}} \times v_a$$

$$\Rightarrow v_o = 10^{-4} \text{ V} \times (-10^{-4}) \frac{\text{A}}{\text{V}} \times 0.1 \text{ V}$$

$$\Rightarrow v_o = -10^{-9} \text{ V} = -1 \text{ nV}$$

Since the question asks for amplitude of $v_o$, $+10^{-9} \text{ V}$ was also accepted as a correct answer.

Alternatively, since we know from Part (a) that $V_A = 10 \text{ V}$ biases the nonlinear element in its linear regime, we can replace the nonlinear element in the op-amp circuit with its equivalent resistance, $r_n$, from its linear regime (where $v_N < -3 \text{ V}$), where

$$r_n = \left( \frac{di_N}{dv_N} \right)^{-1} = 10^{-4} \Omega$$

Now $v_n = i_n r_n$ and, since $v_o = v_n$, we get $v_o = i_n r_n$, where $i_n = -\frac{v_o}{10^{-5}} = -0.1 \text{ V} = -10^{-5} \text{ A}$.

$$\Rightarrow v_o = i_n r_n = -10^{-5} \times 10^{-4} = -10^{-9} \text{ V}$$

[Note: Since units were not specified for $i_N$ in the given $i_N - v_N$ equations, $v_o = \pm10^{-6} \text{ V}$, obtained by assuming $i_N$ as having units of mA in the given $i_N - v_N$ equations, was also accepted as a correct answer.]
Problem 6 [30 points]: Power Electronics

The circuit shown below contains a constant current source (I), a capacitor (C), an inductor (L) and two ideal switches (S1 and S2). Prior to t = 0, both switches are closed as shown, and the capacitor state (v) and inductor state (i) are both zero.

![Circuit Diagram]

a) [5 points] At t = 0, S1 opens while S2 remains closed as shown below. Determine v(t) and i(t) in terms of I, C and L for t ≥ 0.

![Circuit Diagram with Switches Changed]

<table>
<thead>
<tr>
<th>v(t)</th>
<th>( \frac{It}{C} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>i(t)</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ I = C \frac{dv}{dt} \Rightarrow v = \frac{1}{C} \int_0^t I \, dt = \frac{It}{C} \]

\[ 0 = L \frac{di}{dt} \Rightarrow i = \text{constant} = 0 \]
b) [5 points] At $t = T_1$, S1 closes while S2 opens simultaneously as shown below. Determine $v(t)$ and $i(t)$ in terms of $I$, $C$, $L$ and $T_1$ for $t \geq T_1$.

\[
\begin{align*}
\text{v(t)} & \quad \frac{I}{C} \, T_1 \cos \left( \frac{t-T_1}{\sqrt{L/C}} \right) \\
\text{i(t)} & \quad \frac{I}{\sqrt{L/C}} \, T_1 \sin \left( \frac{t-T_1}{\sqrt{L/C}} \right)
\end{align*}
\]

At $t = T_1$, $i = 0$ and $v = \frac{I}{C} T_1$.

For $t \geq T_1$, $v = L \frac{di}{dt}$ and $i = -C \frac{dv}{dt} \Rightarrow$

\[
L C \frac{d^2 V}{dt^2} + V = 0 \quad \Rightarrow \quad V = A_v \cos \left( \frac{t-T_1}{\sqrt{L/C}} \right) + A_s \sin \left( \frac{t-T_1}{\sqrt{L/C}} \right)
\]

and $i = \sqrt{\frac{C}{L}} \left[ A_v \sin \left( \frac{t-T_1}{\sqrt{L/C}} \right) - A_s \cos \left( \frac{t-T_1}{\sqrt{L/C}} \right) \right]$. Initial conditions applied at $t = T_1$ \( \Rightarrow \) \( A_v = \frac{I}{C} T_1 \) and \( A_s = 0 \).
c) [5 points] Let T2 be the first time after T1 at which v(t) reaches 0 V. Determine T2 in terms of I, C, L and T1.

<table>
<thead>
<tr>
<th>T2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T1 + \frac{\pi}{2} \sqrt{LC} )</td>
</tr>
</tbody>
</table>

\[
\cos \left( \frac{t - T1}{\sqrt{LC}} \right) = 0 \quad \Rightarrow \quad \frac{t - T1}{\sqrt{LC}} = \frac{\pi}{2}
\]

\[
\Rightarrow t = T1 + \frac{\pi}{2} \sqrt{LC}
\]
d) [4 points] At T2, S2 closes while S1 remains closed as shown below. Determine $v(t)$ and $i(t)$ in terms of $I$, $C$, $L$, $T1$ and $T2$ for $t \geq T2$.

\[
\begin{array}{c|c}
 v(t) & \\
 \hline
 i(t) & \frac{I}{\sqrt{LC}} \times T1 \\
\end{array}
\]

$S1$ and $S2$ both closed $\Rightarrow V = 0$.

$S2$ closed $\Rightarrow 0 = L \frac{di}{dt} \Rightarrow i = \text{Constant}$

$\Rightarrow i = \frac{I}{\sqrt{LC}} \times T1$
e) [6 points] On the axes provided below, sketch and clearly label \( v(t) \) and \( i(t) \) for \( t \geq 0 \).
f) [5 points] At T3, with T3 > T2, the two switches again cycle through the states as described in parts A and B: (1) S1 opens for a duration T1 while S2 remains closed; (2) at t = T3 + T1 the two switches simultaneously change states. In terms of I, C, L, T1, T2 and T3, determine the value of i when v(t) first reaches 0 V after the switches change states, that is, after t = T3 + T1.

\[
\begin{array}{c|c}
  i & \sqrt{2} \frac{I T1}{\sqrt{L C}} \\
\end{array}
\]

During the first part of each cycle, the capacitor charges to \( \frac{I T1}{C} \). Its energy stored is then \( \frac{C}{2} \left( \frac{I T1}{C} \right)^2 \). During the second part of each cycle, this energy is fully transferred to the inductor, adding to any energy previously stored in the inductor. During the third part of each cycle, the inductor current remains constant. Therefore, after two cycles

\[
\frac{1}{2} L i^2 = 2 \times \frac{C}{2} \left( \frac{I T1}{C} \right)^2 \Rightarrow i = \sqrt{2} \frac{I T1}{\sqrt{L C}}
\]

Inductor Energy

Two Cycles

Capacitor Energy
Problem 7 [20 points]: Electromechanics

A Lorentz-force loudspeaker is actuated by a coil carrying current in the presence of a magnetic field generated by a permanent magnet. The coil is attached to the moving speaker cone while the magnet is attached to the stationary speaker frame. Such a loudspeaker can be modeled by the circuit shown below. The left-hand side models the mechanics of the loudspeaker, representing common velocity as voltage ($u$), and force as current, with each device having its own force and hence current. The mechanics comprise a speaker-cone spring with stiffness $K$ modeled as an inductor, a speaker cone and coil with total mass $M$ modeled as a capacitor, and a damper with viscous damping $B$ modeled as a resistor. Loss in the resistor represents radiated acoustic power. The Lorentz-force electromechanical energy conversion process is modeled by two dependent sources with coefficient $G$ having units of [N/A] and [Vs/m]. The inductance $L$ and resistance $R$ are properties of the coil. The coil terminal variables are $i$ and $v$ at the far right.
a) [10 points] The original loudspeaker model is shown below together with a simplified model. Determine the simplified model parameters $C'$, $L_1'$, $L_2'$, $R_1'$ and $R_2'$ in terms of the original model parameters $B$, $G$, $K$, $L$, $M$ and $R$.

Original model:

Simplified model:

<table>
<thead>
<tr>
<th>$C'$</th>
<th>$L_1'$</th>
<th>$L_2'$</th>
<th>$R_1'$</th>
<th>$R_2'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{M}{G^2}$</td>
<td>$\frac{G^2}{K}$</td>
<td>$L$</td>
<td>$\frac{G^2}{B}$</td>
<td>$R$</td>
</tr>
</tbody>
</table>

$L_2'$ and $R_2'$ is determined by inspection, since $L$ and $R$ are already electrical terms.

The rest can be determined by finding the equivalent passive element network of $Gu$

let $V_g$ be the voltage across $Gu$, so $V_g = Gu$

In s.s.s. $U = G i \left( \frac{1}{\frac{k}{s} + sM + B} \right)$

$V_g = G^2 i \left( \frac{1}{\frac{k}{s} + sM + B} \right)$

$\frac{V_g}{i} = Z = \frac{G^2}{\frac{k}{s} + sM + B}$

since the simplified model is a parallel RLC, we can convert $Z$ into an admittance

$\frac{1}{\frac{1}{Z}} = \frac{1}{\frac{k}{s}G^2} + \frac{1}{sM} + \frac{1}{B}$

admittances in parallel add, we can read out the terms here.
b) [10 points] Let the loudspeaker be driven in the sinusoidal steady state with $v(t) = V \cos(\omega t)$. In this case, the loudspeaker velocity $u(t)$ takes the form $u(t) = U \cos(\omega t + \phi)$. Determine $U$ and $\phi$ in terms of $V$, $\omega$ and any of the original or simplified model parameters. To simplify this problem, assume the original resistance $R$ is negligible such that $R = 0$.

$$
\begin{align*}
U & = \frac{G/L}{\sqrt{\left(\frac{G}{L}^2 + k - \omega^2 M\right)^2 + (\omega B)^2}} V \\
\phi & = \arctan\left(-\frac{\omega B}{\frac{G}{L}^2 + k - \omega^2 M}\right)
\end{align*}
$$

Use previous results, the impedance approach, and apply voltage divider.

$$
\begin{align*}
\frac{Z}{Z+SL^2} \tilde{V}(s) &= G_T \tilde{u}(s) \Rightarrow \frac{1}{G} \frac{Z}{Z+SL} \tilde{V}(s) = \tilde{u}(s) \\
\frac{G_T^2}{\frac{k}{s} + sM + B} \tilde{u}(s) &= \tilde{v}(s) \Rightarrow \frac{G_T \tilde{V}(s)}{G^2 + Lk + s^2LM + sLB} = \tilde{u}(s)
\end{align*}
$$

$$
\begin{align*}
\rightarrow \quad \tilde{v}(s) &= \tilde{V}(s) = \tilde{u}(s) \\
\tilde{v}(\omega) &= \tilde{u}(\omega) \\
U &= \frac{G/L}{\sqrt{\left(\frac{G}{L}^2 + k - \omega^2 M\right)^2 + (\omega B)^2}} V \\
\phi &= \arctan\left(-\frac{\omega B}{\frac{G}{L}^2 + k - \omega^2 M}\right)
\end{align*}
$$