Lecture 3 - **Node Analysis**

February 12, 2019

**Contents:**
1. Review: series and parallel simplification; V and I dividers
2. Circuit analysis through node method

**Reading Assignment:**
Agarwal and Lang, Ch. 3, § 3.1-3.3 (3.3.1, 3.3.2)

**Handouts:**
Lecture 3 notes

**Announcements:**

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**Review:** We have more useful simplification and analysis tricks:

- **Resistors in series**
  - Share the same current
  
- **Resistors in parallel**
  - Share the same voltage

\[ R_T = R_1 + R_2 + \ldots + R_N \]

\[ V_T = V_1 + V_2 + \ldots + V_N \]

- **Voltage divider:**
  \[ V_K = \frac{R_K}{R_1 + \ldots + R_N} V_T \]

- **Current divider:**
  \[ i_K = \frac{G_K}{G_1 + G_2 + \ldots + G_N} i_T \]

- **Conductance (resistors in parallel):**
  \[ G_K = \frac{1}{R_K} \]

- **Conductance (resistors in series):**
  \[ 1 = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + 1 \]

\[ \frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_N} \]
Series and parallel

Which are true?
1. R1 is in series with R3
2. R1 is in parallel with R2
3. R2 is in series with R4
4. R3 is in parallel with R4
5. (R1+R3) is in parallel with (R2+R4)

Which are true?
1. R1 is in series with R3
2. R3 is in parallel with R4

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Series and parallel

What is $v_2$?


Resistor Divider:

\[ V_2 = i_2 R_2 = i_T R_2 \]

\[ V_2 = \frac{V_0}{R_1 + R_2} \]

\[ V_2 = \frac{R_2}{R_1 + R_2} \cdot V_0 \]

\[ i_1 = \frac{V_0}{R_1} \cdot \frac{R_2}{R_1 + R_2} \]

Voltage Divider:

\[ V_2 = V_1 = V_T \]

\[ i_1 = \frac{V_T}{R_1} = \frac{1}{R_1} = \frac{G_1}{G_1 + G_2} = \frac{1}{R_2} \]

Current Divider:

Useful Approximations / Simplifications:

- Series: large resistor dominates.
- Voltage divider: large resistor takes up most of the voltage.
- Parallel: smallest resistor dominates.
- Current divider: smallest resistor takes up most of the current.
Voltage divider

- How to pick $R$ to maximize $v_L$?

Sensor
Amplifier
Teensy
etc.

Headphones
Motor
etc.

Current divider

- What is $i_2$?

10 mA

$\begin{align*}
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\end{align*}$
Resistive dividers

Roughly:
- What is overall resistance?
- Which resistor takes up most voltage?
- What is voltage across 100 kΩ resistor?
- What is voltage across 100 Ω resistor?

Series and parallel

Roughly:
- What is overall resistance?
- Which resistor takes up most current?
- What is current through 100 Ω resistor?
Example 1

Voltage divider: \[ V_3 = \frac{R_{2//R_3}}{R_1 + R_{2//R_3}} V_0 \]

\[ i_3 = \frac{V_3}{R_3} = \frac{1}{R_3 \left( \frac{R_{2//R_3}}{R_1 + R_{2//R_3}} \right)} \]

\[ V_0 = i_3 \]

Example 2.

Current divider:

\[ i_3 = -\frac{R_1}{R_1 + (R_2 + R_3)} I_0 \]

\[ V_3 = i_3 R_3 = \frac{-I_0 R_1 R_2}{R_1 + R_2 + R_3} \]
Node Analysis

Five-step process:
1. Select reference node ("ground") from which voltages are measured. Effectively, set the reference node at 0 V.

2. Label voltages of all remaining unsourced nodes with respect to ground. These are the primary unknowns.

3. Label branch currents and write KCL for all unsourced nodes, substituting device laws.
   These equations are now in terms of the node voltages.

4. Solve KCL equations for node voltages.

5. Solve for branch voltages and currents, as needed.

Focus on finding $N-1$ node voltages, + back solve for all else?
We already know \( e_3 = V_o \).

We solve for \( e_1, e_2 \) using KCL.

Q: What if we mislabeled the direction of currents? And Cos...

\[ e_1 \text{ node} \rightarrow (V_0 - e_1) \bar{G}_1 - e_1 \bar{G}_2 - (e_1 - e_2) \bar{G}_3 = 0 \]
\[ (i_1 - i_2 - i_3 = 0) \]
\[ e_2 \text{ node} \rightarrow (e_1 - e_2) \bar{G}_3 + (V_0 - e_2) \bar{G}_4 - e_2 \bar{G}_5 + I = 0. \]

These are linear in \( e_1, e_2, V_0, I \). Let us set up as a matrix solution!

\[
\begin{bmatrix}
\bar{G}_1 + \bar{G}_2 + \bar{G}_3 & -\bar{G}_3 \\
-\bar{G}_3 & \bar{G}_3 + \bar{G}_4 + \bar{G}_5
\end{bmatrix}
\begin{bmatrix}
\bar{e}_1 \\
\bar{e}_2
\end{bmatrix}
= 
\begin{bmatrix}
\bar{G}_1 V_0 \\
\bar{G}_4 V_0 + I
\end{bmatrix}
\]
\[ G \cdot e = S \quad \rightarrow \quad e = G^{-1} \cdot S \]

And we get:

\[
\begin{bmatrix}
  e_1 \\
  e_2
\end{bmatrix} = \frac{\begin{bmatrix}
  G_3 + G_4 + G_5 \\
  G_3 + G_4 + G_5
\end{bmatrix} (G_3 + G_4) + G_5 - G_2}{(G_1 + G_2 + G_5) (G_3 + G_4 + G_5) - G_2^2}
\begin{bmatrix}
  G_3 + G_4 + G_5 \\
  G_3 + G_4 + G_5 + G_3 \\
  G_3 + G_4 + G_5
\end{bmatrix} \begin{bmatrix}
  G_1 + G_2 + G_5 \\
  G_1 + G_2 + G_5
\end{bmatrix}
\]

Or

\[
e_1 = \frac{(G_3 + G_4 + G_5) (G_3 V_0) + (G_3 + G_4 + G_5) I_1}{G_1 G_3 + G_1 G_4 + G_1 G_5 + G_2 G_3 + G_2 G_4 + G_2 G_5 + G_3 G_3 + G_3 G_4 + G_3 G_5}
\]

\[
e_2 = \frac{(G_3 + G_4 + G_5) (G_3 V_0) + (G_1 + G_2 + G_5) (G_3 V_0 + I_1)}{G_1 G_3 + G_1 G_4 + G_1 G_5 + G_2 G_3 + G_2 G_4 + G_2 G_5 + G_3 G_3 + G_3 G_4 + G_3 G_5}
\]

(same denominator)

Notice:

- linear in \( V_0 \) and \( I_1 \)
- no negative terms in denominator
- units work out

Notes:
1. Their method always works (i.e., analyzing solution for "linear" components) (Note: Use "superscale" for floating values)
2. We can easily solve for other variables:

\[ i_3 = (e_1 - e_2) G_3, \quad V_3 = e_1 - e_2 \]
3. Relationship between form of the network and the conductance matrix and source vector:

\[ [i, j] \rightarrow \text{Sum of conductances connected to node } i \]

\[ [i, j] \rightarrow \text{Negative sum of conductances between nodes } i, j \]

\[ [i] \rightarrow \text{Sum of independent currents into node } i \]

plus independent voltage; that time, the conductances linking them to node \( i \).

- We can go directly from topology to matrix.
- We can go backwards (synthesize a network that represents a set of equations).

### Numerical example

![Circuit Diagram]

- \( R_1 = 8.2 \text{ k}\Omega \)
- \( R_2 = 3.9 \text{ k}\Omega \)
- \( R_3 = 3.9 \text{ k}\Omega \)
- \( R_4 = 8.2 \text{ k}\Omega \)
- \( R_5 = 1.5 \text{ k}\Omega \)
- \( V_0 = 3 \text{ V} \)
- \( I_1 = 0 \text{ A} \)

- \( e_1 = 1.38 \text{ V} \)
- \( e_2 = 1.62 \text{ V} \)

\( i_0 = 0.57 \text{ mA} \)

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Application: Machine learning

Inputs \rightarrow Neurons \rightarrow Outputs

\[ y_1 = x_1 y_1 + x_2 y_2 + x_3 y_3 \]
\[ y_2 = x_4 y_1 + x_5 y_2 + x_6 y_3 \]
\[ y_3 = x_7 y_1 + x_8 y_2 + x_9 y_3 \]

\[ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leftrightarrow \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

How to do matrix multiplication in a very efficient way?

\[ R_1 = \frac{1}{x_1} \]
\[ R_2 = \frac{1}{x_2} \]
\[ R_3 = \frac{1}{x_3} \]
\[ R_4 = \frac{1}{x_4} \]
\[ R_5 = \frac{1}{x_5} \]
\[ R_6 = \frac{1}{x_6} \]
\[ R_7 = \frac{1}{x_7} \]
\[ R_8 = \frac{1}{x_8} \]
\[ R_9 = \frac{1}{x_9} \]

\[ y = y_1 x_2 + y_2 x_8 + y_3 x_9 \]