Lecture 14 - Impedance

April 4, 2019

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Reading Assignment:
Agarwal and Lang, Ch. 13 (§ § 13.3, 13.4)

Handouts:
Lecture 14 notes

Announcements:
1. Please do prelab and read lab description in advance of tomorrow’s lab.
1. Review of method of complex exponentials

Interested in analyzing response of a linear system to a sinusoidal steady state (SSS) drive:

\[ \cos \omega t \rightarrow LS \rightarrow V_C \cos(\omega t + \phi) \]

New abstract problem in the mathematical domain.
- Fast solution.
- Can recover physical solution by:

\[ M_1 = \Re[\tilde{M}] \]
2. Concept of impedance and admittance

In the real world, device equations relate voltage and current in a branch. For example:

\[ v_R = R \dot{i}_R \quad i_C = C \frac{dv_C}{dt} \quad v_L = L \frac{di_L}{dt} \]

Same relations apply for complex exponential drives:

\[ \tilde{v}_R = \tilde{R} \tilde{i}_R \quad \tilde{i}_C = C \frac{d\tilde{v}_C}{dt} \quad \tilde{v}_L = L \frac{d\tilde{i}_L}{dt} \]

where...

\[ \tilde{v}_N = \tilde{V}_n e^{j\omega t} \quad \tilde{i}_N = \tilde{I}_n e^{j\omega t} \]
Device equations for complex exponential signals

• Resistor:

Device equation:

\[ v_R = R i_R \]

For complex exponential drive:

\[ \tilde{v}_R = R \tilde{i}_R \Rightarrow \tilde{V}_r e^{j\omega t} = R \tilde{I}_r e^{j\omega t} \]

Or:

\[ \tilde{V}_r = R \tilde{I}_r \]

Since I can dispose of \( e^{j\omega t} \) term… We can formulate these device equations in terms of the complex amplitudes only

Linear relationship between complex voltage amplitude and complex current amplitude in resistor.
• Capacitor:
Device equation:

\[ i_C = C \frac{dv_C}{dt} \]

For complex exponential drive:

\[ \tilde{i}_C = C \frac{d\tilde{v}_C}{dt} \Rightarrow \tilde{I}_c e^{j\omega t} = C j\omega \tilde{V}_c e^{j\omega t} \]

Or:

\[ \tilde{I}_c = j\omega C \tilde{V}_c \]

Linear relationship between complex voltage amplitude and complex current amplitude in capacitor...

... just like in resistor!
• Inductor:

Device equation:

\[ v_L = L \frac{di_L}{dt} \]

For complex exponential drive:

\[ \tilde{v}_L = L \frac{\tilde{d}i_L}{dt} \Rightarrow \tilde{V}_i e^{j\omega t} = L j \omega \tilde{I}_i e^{j\omega t} \]

Or:

\[ \tilde{V}_i = j \omega L \tilde{I}_i \]

Linear relationship between complex voltage amplitude and complex current amplitude in inductor...

... also like in resistor!
Impedance and admittance

*Define impedance* as the ratio of complex voltage amplitude and complex current amplitude in a device:

\[ Z = \frac{\tilde{V}_n}{\tilde{I}_n} \]

Similarly, *define admittance* as the ratio of complex current amplitude and complex voltage amplitude in a device:

\[ Y = \frac{\tilde{I}_n}{\tilde{V}_n} = \frac{1}{Z} \]

In general, Z and Y are complex:

\[ Z = R + jX \quad \text{reactance} \]

\[ Y = G + jB \quad \text{susceptance} \]

\[ \text{resistance} \]

\[ \text{conductance} \]
Impedance and admittance of resistor, capacitor and inductor

- Resistor:
  \[ \tilde{V}_r = R\tilde{I}_r \quad \rightarrow \quad Z_R = R \quad Y_R = \frac{1}{R} = G \]

- Capacitor:
  \[ \tilde{I}_c = j\omega C\tilde{V}_c \quad \rightarrow \quad Z_C = \frac{1}{j\omega C} \quad Y_C = j\omega C \]

- Inductor:
  \[ \tilde{V}_l = j\omega L\tilde{I}_l \quad \rightarrow \quad Z_L = j\omega L \quad Y_L = \frac{1}{j\omega L} \]

But... what does this really mean??!!

→ If fed with SSS current, one can easily get the voltage waveform across the element
• Resistor:
  - Impedance:
    \[ Z_R = R \]
  
  - Apply:
    \[ i_R = I_r \cos \omega t = \text{Re}[I_r e^{j\omega t}] \]
  
  - Voltage across resistor:
    \[ v_R = \text{Re}[\tilde{V}_r e^{j\omega t}] = \text{Re}[RI_r e^{j\omega t}] = RI_r \cos \omega t \]

\[ i_R(t) \]
\[ I_r \]
\[ 0 \]
\[ t \]

\[ v_R(t) \]
\[ V_r = RI_r \]
\[ 0 \]
\[ t \]
• Capacitor:
  - Impedance:
    \[ Z_C = \frac{1}{j\omega C} \]
  - Apply:
    \[ i_C = I_c \cos \omega t = Re[I_c e^{j\omega t}] \]
  - Voltage across capacitor:
    \[ v_C = Re[\tilde{V}_c e^{j\omega t}] = Re[\frac{I_c}{j\omega C} e^{j\omega t}] = Re[\frac{I_c}{\omega C} e^{j(\omega t - \frac{\pi}{2})}] \]
    \[ = \frac{I_c}{\omega C} \cos(\omega t - \frac{\pi}{2}) = \frac{I_c}{\omega C} \sin \omega t \]
• Inductor:
  
  - Impedance:
    
    \[ Z_L = j\omega L \]
  
  - Apply:
    
    \[ i_L = I_l \cos \omega t = Re[I_l e^{j\omega t}] \]
  
  - Voltage across inductor:
    
    \[ v_L = Re[\tilde{V}_l e^{j\omega t}] = Re[j\omega L I_l e^{j\omega t}] = Re[\omega L I_l e^{j(\omega t + \frac{\pi}{2})}] \]
    
    \[ = \omega L I_l \cos(\omega t + \frac{\pi}{2}) = -\omega L I_l \sin \omega t \]
Impedance in the frequency domain:

- **Resistor:**

\[ Z_R = R \quad |Z_R| = R \quad \phi_{Z_R} = 0 \]

Plot of impedance magnitude and angle vs. angular frequency:

Resistor behavior independent of frequency.
• Inductor:

\[ Z_L = j\omega L \quad |Z_L| = \omega L \quad \phi_{Z_L} = \frac{\pi}{2} \]

Note:

\[ \log |Z_L| = \log(L) + \log(\omega) \]

Plot of impedance magnitude and angle vs. angular frequency:

- At low frequency, L shorts out
- At high frequency, L opens up
• Capacitor:

\[ Z_C = \frac{1}{j\omega C} \quad |Z_C| = \frac{1}{\omega C} \quad \phi_{Z_C} = -\frac{\pi}{2} \]

Note:

\[ \log |Z_C| = -\log(C) - \log(\omega) \]

Plot of impedance magnitude and angle vs. angular frequency:

- At low frequency, C opens up
- At high frequency, C shorts out
3. Impedance method to analyze SSS response of complex circuits

To solve a circuit, we need:
- KCL on all relevant loops
- KVL on all relevant nodes
- Device equations

KCL and KVL reflect the circuit topology.

For SSS situations, want to use complex amplitudes of voltage and current signals.

How do KCL and KVL become when expressed in terms of complex amplitudes?
• KVL:

\[ \tilde{V}_1 e^{j\omega t} + \tilde{V}_2 e^{j\omega t} + \tilde{V}_3 e^{j\omega t} = 0 \]

Simplifies to:

\[ \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 = 0 \]

• KCL:

\[ \tilde{I}_1 e^{j\omega t} + \tilde{I}_2 e^{j\omega t} + \tilde{I}_3 e^{j\omega t} = 0 \]

Simplifies to:

\[ \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 = 0 \]
To summarize, when working with SSS drives described by complex exponentials:

- the complex amplitudes of voltage and current fulfill KCL and KVL

\[ \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 = 0 \quad \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 = 0 \]

- the device equations in terms of complex amplitudes are linear

\[ \tilde{V}_r = R\tilde{I}_r \quad \tilde{V}_l = j\omega L\tilde{I}_l \quad \tilde{I}_c = j\omega C\tilde{V}_c \]

→ can solve for complex amplitudes using the simple methods developed for purely resistive networks

→ no need to formulate nor solve a differential equation!
Impedance method

\[ v_{IN}(t) = V_{in} \cos \omega t \]

1. Express SSS drive in terms of complex exponential.

2. Substitute elements in circuit by their equivalent impedance; express current and voltages in terms of their complex amplitudes.

3. Use techniques developed for purely resistive networks to solve for complex amplitudes in circuit.

4. Derive real solution from complex exponential solution.
1. Express SSS drive in terms of complex exponential.

For:

\[ v_{IN}(t) = V_{in} \cos \omega t \]

We have:

\[ \tilde{v}_{IN} = V_{in}e^{j\omega t} \]
2. Substitute elements in circuit by their equivalent impedance; express current and voltages in terms of their complex amplitudes.
3. Use techniques developed for purely resistive networks to solve for complex amplitudes in circuit.

Easy! → Voltage divider:

\[ \tilde{V}_c = \frac{1}{R + \frac{1}{j\omega C}} V_{in} = \frac{V_{in}}{1 + j\omega RC} \]
4. Derive real solution from complex exponential solution.

\[ \tilde{V}_c = \frac{V_{in}}{1 + j\omega RC} \]

Rewrite:

\[ \tilde{V}_c = \frac{V_{in}}{\sqrt{1 + \omega^2 R^2 C'^2}} e^{-j\tan^{-1}(\omega RC)} \]

From which we get:

\[
\begin{align*}
\quad v_C(t) &= \quad Re[\tilde{V}_c e^{j\omega t}] \\
&= \frac{V_{in}}{\sqrt{1 + \omega^2 R^2 C'^2}} \cos[\omega t - \tan^{-1}(\omega RC)]
\end{align*}
\]

Done! Did not have to solve (or even formulate) a differential equation!
4. A note about Pre-Lab 8...

**Voltage gain (dB) = 20 \times \frac{V_{out}(f)}{V_{in}(f)}**

3 dB or 0.707 from original value

\[ Q = \frac{f_0}{BW} \]
Summary

• Impedance: ratio of complex voltage amplitude to complex current amplitude in a device when driven with complex exponential signal.

• Impedance of resistor:

\[ Z_R = R \]

• Impedance of capacitor:

\[ Z_C = \frac{1}{j\omega C} \]

• Impedance of inductor:

\[ Z_L = j\omega L \]

• Using impedances, can solve SSS response of complex linear networks as if purely resistive.