6.002 Recitation - Spring 2019  
Farnaz Niroui, Rm 12-5007, fniroui@mit.edu

Outline  
RLC Circuits  
Quality Factor

Review from Last Two Weeks

RC Circuits

In a capacitor
\[ \frac{\psi_C(t)}{I} = \frac{V_C(t)}{I} \]
\[ i_C(t) = \frac{d\psi_C(t)}{dt} = C \frac{dV_C(t)}{dt} \]

\[ i_R \]
\[ R \]
\[ v_C(t) \]
\[ v_s(t) \]

\[ v_C(t) = v_s(1 - e^{-t/T}) \text{ where } T = RC \]
\[ i_C(t) = \frac{v_s}{R} e^{-t/T} \]

\[ V_C(t) \]
\[ v_s \]
\[ i_C(t) \]

RL Circuits

In an inductor:
\[ L \frac{d^2i_L(t)}{dt^2} + Ri_L(t) = V_L(t) \]
\[ i_R \]
\[ R \]
\[ v_L(t) \]
\[ v_s(t) \]

\[ i_L(t) = \frac{v_s}{R} (1 - e^{-t/T}) \text{ where } T = L/R \]
\[ V_L(t) = v_s e^{-t/T} \]

\[ V_L(t) \]
\[ v_s \]
\[ i_L(t) \]
Undriven, Series RLC Circuit

\[ \begin{align*}
\text{Node 1:} & \quad C \frac{dV_1(t)}{dt} + \frac{V_1(t) - V_2(t)}{R} = 0 \\
\text{Node 2:} & \quad \frac{V_2(t) - V_1(t)}{R} + \frac{1}{L} \int_0^t V_2(t') \, dt' = 0
\end{align*} \]

Use equation (1) to find \( V_2(t) \) and then use that in (2):

\[ \begin{align*}
V_2(t) &= V_1(t) + RC \frac{dV_1(t)}{dt} \\
\text{using (2)} & \Rightarrow \quad V_2(t) - V_1(t) + \frac{R}{L} \int_0^t V_2(t') \, dt' = 0 \\
\frac{dV_2(t)}{dt} - \frac{dV_1(t)}{dt} + \frac{R}{L} V_2(t) &= 0 \\
\frac{d}{dt} \left( V_1(t) + RC \frac{dV_1(t)}{dt} \right) - \frac{dV_1(t)}{dt} + \frac{R}{L} \left( V_1(t) + RC \frac{dV_1(t)}{dt} \right) &= 0 \\
RC \frac{d^2V_1(t)}{dt^2} + \frac{R}{L} V_1(t) + \frac{R^2C}{L} \frac{dV_1(t)}{dt} &= 0
\end{align*} \]

Second order differential equation
\[ \frac{d^2 V_i(t)}{dt^2} + \frac{R}{L} \frac{dV_i(t)}{dt} + \frac{1}{LC} V_i(t) = 0 \]

Let \( \alpha = \frac{R}{2L} \) and \( \omega_c = \frac{1}{\sqrt{LC}} \)

\[ \frac{d^2 V_i(t)}{dt^2} + 2\alpha \frac{dV_i(t)}{dt} + \omega_c^2 V_i(t) = 0 \]

Solution to Homogeneous Equation is of the form \( e^{st} \)

Characteristic equation: \( s^2 + 2\alpha s + \omega_c^2 = 0 \)

\[ s = -\alpha \pm j\omega_d = -\alpha \pm j\sqrt{\omega_c^2 - \alpha^2} = -\alpha \pm j\omega_d \]

\[ \omega_d = \sqrt{\omega_c^2 - \alpha^2} \]

\[ V_i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \]

\[ s_1 = -\alpha + j\omega_d \text{ and } s_2 = -\alpha - j\omega_d \]

Find \( A_1 \) and \( A_2 \) using the initial conditions. What is \( V_i(0), \) and \( \frac{dV_i(0)}{dt} \)?

1. \( V_C(0) = V_i(0) \Rightarrow V_i(0) = V_C(0) \)

2. \( i_L(1) = -C \frac{dV_i(t)}{dt} \Rightarrow i_L(0) = -C \frac{dV_i(0)}{dt} \Rightarrow \frac{dV_i(0)}{dt} = -\frac{1}{C} i_L(0) \)

\[ V_i(0) = A_1 + A_2 = V_C(0) \]

\[ \frac{dV_i(0)}{dt} = A_1 s_1 + A_2 s_2 = -\frac{1}{C} i_L(0) \]

\[ A_1 = \frac{C s_2 V_C(0) + i_L(0)}{C(s_2 - s_1)} \]

\[ A_2 = \frac{C s_1 V_C(0) + i_L(0)}{C(s_1 - s_2)} \]

\[ V_i(t) = \frac{C s_2 V_C(0) + i_L(0)}{C(s_2 - s_1)} e^{s_1 t} + \frac{C s_1 V_C(0) + i_L(0)}{C(s_1 - s_2)} e^{s_2 t} \]

\[ \begin{cases} V_C(t) = V_i(t) = \frac{C s_2 V_C(0) + i_L(0)}{C(s_2 - s_1)} e^{s_1 t} + \frac{C s_1 V_C(0) + i_L(0)}{C(s_1 - s_2)} e^{s_2 t} \\ i_L(t) = -s_1 \frac{C s_2 V_C(0) + i_L(0)}{(s_2 - s_1)} e^{s_1 t} - s_2 \frac{C s_1 V_C(0) + i_L(0)}{(s_1 - s_2)} e^{s_2 t} \end{cases} \]
\[ S = -\alpha \pm \sqrt{\alpha^2 - \omega_c^2} \]

Three Scenarios

1. \( \omega_c > \alpha \) under damped: Two damped oscillator solutions
   \[ V_i(t) = e^{-\alpha t} \left[ A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t) \right] \]
   Quality factor: \[ Q = \frac{\omega_c}{2\alpha} \]
   \[ \omega_d = \frac{1}{\sqrt{L C}} \] undamped natural frequency
   \[ \alpha = \text{damping factor} \]
   \( \omega_c \) = damped natural frequency

\[ \frac{1}{\omega_c} \cdot \frac{1}{R} = \frac{1}{R} \sqrt{\frac{L}{C}} \]

\( \alpha \) is large if \( \alpha \) is small relative to \( \omega_c \), characteristic of under damped system - circuit will oscillate for a long time.

2. \( \omega_c = \alpha \) critically damped - Two decaying solutions (repeated root)
   \[ V_i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} \]

3. \( \omega_c < \alpha \) over damped - Two decaying exponentials
   \[ V_i(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} \]

---

**Diagram:**

- **Left Diagram:**
  - \( v_c(0) \)
  - Over damped
  - Critically damped
  - Under damped

- **Right Diagram:**
  - \( i_c \)
  - Under damped
  - Critically damped
  - Over damped
Agenda
- Parallel RLC Source Example
  - Step Response
  - Impulse response

Using KCL at node $V_c$:

$$\frac{V_c}{R} + \frac{1}{L} \int_0^t V_c(t') dt' + C \frac{dV_c}{dt} - 2V_n = 0$$

Differentiating again:

$$\frac{1}{R} \frac{dV_c}{dt} + \frac{1}{L} V_c + C \frac{d^2V_c}{dt^2} = \frac{dV_n}{dt}$$

$$\frac{d^2V_c}{dt^2} + \frac{1}{RC} \frac{dV_c}{dt} + \frac{V_c}{LC} = \frac{1}{C} \frac{dV_n}{dt}$$

Let $\alpha = \frac{1}{2RC}$ and $\omega_n^2 = \frac{1}{LC}$
\[
\frac{d^2 V_c}{dt^2} + 2\alpha \frac{dV_c}{dt} + \omega_n^2 = \frac{1}{C} \frac{dI_e}{dt}
\]

Note that we could also obtain the same form for \( L \).

Solution to Homogeneous Equation is of the form \( e^{st} \)

\[
s^2 + 2\alpha s + \omega_n^2 = 0
\]

\[
\Rightarrow s = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2} = -\alpha \pm j\sqrt{\omega_n^2 - \alpha^2}
\]

\[
= \alpha + j\omega d \quad \omega d = \sqrt{\omega_n^2 - \alpha^2}
\]

Three Scenarios for

1. \( \omega_n < \alpha \) Overdamped - Two decaying exponentials
   \[ V_{ch} = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad s_1, s_2 > 0 \]

2. \( \omega_n = \alpha \) Critically Damped - Two decaying solutions
   \[ V_{ch} = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad s > 0 \]

3. \( \omega_n > \alpha \) Under Damped - Two damped oscillator solutions
   \[ V_{ch} = e^{-\alpha t} \left[ A \cos(\omega d t) + B \sin(\omega d t) \right] \]

\[ V_c(t) = V_{cp} + V_{ch} \]
Step Response (No Energy Stored in Inductor and Capacitor at t=0)

\[ V_c(t) = V_{cH}(t) + V_{cp} = V_{cp} + \text{Homogenous Solution} \]

Let us focus on undamped damped case:

\[ V_c(t) = V_{cp} + e^{-\frac{t}{R}} \left[ A_0 \cos(\omega t) + A_1 \sin(\omega t) \right] \]

What would \( V_{cp} \) be? = \( \text{Vfinal} \)

Let us use intuition:

Long term, the inductor is a short - \( Lc(t) = I_0 \Rightarrow V_{cp} = 0 \)

The capacitor is an open circuit.

\( V_{cp} = 0 \)

How about the initial conditions:

Short term, inductor is an open circuit.

Short term, capacitor is a short circuit \( \Rightarrow I_0 = \frac{dV_{cp}}{dt} \)

No energy stored:

\( E_{cp} = \frac{1}{2} C V_c^2 \Rightarrow V_{c0} = 0 \)

\( E_{ind} = \frac{1}{2} L I_0^2 \Rightarrow I_0 = 0 \)
\[ V_c(t) = e^{-at}(A_c \cos(wd) + A_s \sin(wd)) \]

\[ V_c(0) = 0 = A_c \implies A_c = 0 \]

\[ V_c(t) = e^{-at}A_s \sin(wd) \]

\[ \frac{dV_c(t)}{dt} = \left[ -ae^{-at}A_s \sin(wd) + e^{-at}wdA_s \cos(wd) \right] \]

\[ \frac{d^2V_c(t)}{dt^2} = I_s = CA_s wd \]

\[ A_s = \frac{I_s}{wdC} \]

\[ V_c(t) = \frac{I_s}{wdC}e^{-at} \sin(wdt) = e^{-at} \frac{I_s}{wdC} \sin(wdt) \]

5. **Impulse Response** (Again, no energy stored in inductor or capacitor)

Using intuition, we should expect that

1. \[ V_a(t) \to 0 \text{ as } t \to \infty \]
   - Long term inductor is a short \[ V_c(t \to \infty) = 0 \]

\[ V_c(\infty) = 0 = V_{cp} \]
How about initial conditions?

1. Short term \( L \) is an open cct; \( C \) is a short cct.

2. No energy was stored in capacitor or inductor
   
   \[ E_c = \frac{1}{2} C \left( \frac{U_c}{2} \right)^2 \implies V_c(0^-) = 0 \]
   
   \[ E_L = \frac{1}{2} L (I_2)^2 \implies I_L(0^-) = 0 \]

   The net result of the impulse is that the charge \( Q_0 \) is dumped in the capacitor resulting in a
   
an initial voltage for the capacitor of \( Q_0/C \)

   \[
   -C \frac{dU_c}{dt} = Q_0 \delta(t) \implies \int_{0^-}^{0^+} C \frac{dU_c}{dt} \, dt = \int_{0^-}^{0^+} Q_0 \delta(t) \, dt
   \]

   \[
   C \left( V_c(0^+) - V_c(0^-) \right) = Q_0 \implies V_c(0^+) = \frac{Q_0}{C}
   \]

   Since the inductor is an open circuit \( L_L(0^+) = (I_L(0^-) = 0 \)

   Thus our two initial conditions are

   \[ U_c(0^+) = \frac{Q_0}{C} ; \quad L_L(0^+) = 0 \]

   Thus

   \[ U_c(t) = e^{-at} \left[ A_c \cos(\omega t) + A_s \sin(\omega t) \right] \]

   \[ U_c(0^+) = A_c = \frac{Q_0}{C} \]
The next part of the problem is to find \( A_s \).

We shall use \( L_L(0^+) = 0 \). This approach is messy.

\[
L_L(t) = \frac{1}{L} \int V_c(t) \, dt = \frac{1}{L} \int e^{-at} \left[ A_e \cos(\omega d t) + A_s \sin(\omega d t) \right] \, dt
\]

\[
L_L(t) = \frac{L}{a^2 + \omega^2 d^2} \left[ A_e (-\omega \cos(\omega d t) + \omega d \sin(\omega d t)) - A_s (\omega \sin(\omega d t) + \omega d \cos(\omega d t)) \right]
\]

\[
= \frac{L}{a^2 + \omega^2 d^2} \left[ A_e (\omega d \sin(\omega d t) - \omega \cos(\omega d t)) - A_s (\omega \sin(\omega d t) + \omega d \cos(\omega d t)) \right]
\]

\[
L_L(0^+) = \frac{L}{a^2 + \omega^2 d^2} \left[ -\omega A_e - \omega d A_s \right] = 0 \Rightarrow A_s = -\frac{\omega}{\omega_d} A_e
\]

\[
A_e = \frac{Q_0}{C} \quad A_s = -\frac{\omega}{\omega_d} \frac{Q_0}{C}
\]

\[
V_c(t) = e^{-at} \left[ \frac{Q_0}{C} \cos(\omega d t) - \frac{\omega}{\omega_d} \frac{Q_0}{C} \sin(\omega d t) \right]
\]

\[
= \frac{Q_0 e^{-at}}{\omega d C} \left[ \omega d \cos(\omega d t) - \omega \sin(\omega d t) \right]
\]

\[
V_c = \frac{Q_0}{\omega d C} e^{-at} \left[ \omega d \cos(\omega d t) - \omega \sin(\omega d t) \right]
\]

**Impulse Response**